On Length Minimizing Curves with Distortion Thickness Bounded Below and Distortion Bounded Above

by

Chad A.S. Mullikin

(Under the direction of Jason Cantarella)

Abstract

The distortion of a curve is the supremum, taken over distinct pairs of points of the curve, of the ratio of arclength to spatial distance between the points. Gromov asked in 1981 whether a curve in every knot type can be constructed with distortion less than a universal constant C. Answering Gromov's question seems to require the construction of lower bounds on the distortion of knots in terms of some topological invariant. We attempt to make such bounds easier to prove by showing that points with high distortion are very common on curves of minimum length in the set of curves in a given knot type with distortion bounded above and distortion thickness bounded below.

Index words: Knot Theory, Knot Energy, Gromov's Distortion, Ropelength

ON LENGTH MINIMIZING CURVES WITH DISTORTION THICKNESS BOUNDED BELOW AND DISTORTION BOUNDED ABOVE

by

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DEDICATION

I would like to offer this dedication to many people. First on this list is my son Drake, who has been very patient with his father. Drake acquired his first tooth while this was being written. So, to him and his first tooth, I dedicate the Dragon's tooth curve $\mathcal{D}\colon [0, 6\pi/4 + \pi r/2] \longrightarrow \mathbb{R}^2$

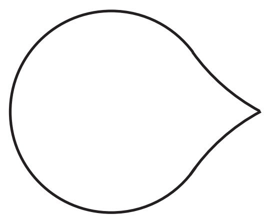

Figure 1: The Dragon's Tooth curve.

$\mathcal{D}(s)$ has the following arclength parametrization:

$$\mathcal{D}(s)\colon = \begin{cases} \left(r\cos\left(\frac{2s+\pi r}{2r}\right) + \frac{(r+1)\sqrt{2}}{2}, r\sin\left(\frac{2s+\pi r}{2r}\right) - \frac{(r+1)\sqrt{2}}{2}\right), & s \in [0, \frac{\pi r}{4}] \\ \left(\cos\left(\frac{4s-\pi(r-1)}{4}\right), -\sin\left(\frac{4s-\pi(r-1)}{4}\right)\right), & s \in [\frac{\pi r}{4}, \frac{\pi r+6\pi}{4}] \\ \left(r\cos\left(\frac{2s+(2r-3)\pi}{2r}\right) + \frac{r\sqrt{2}+\sqrt{2}}{2}, r\sin\left(\frac{2s+(2r-3)\pi}{2r}\right) + \frac{r\sqrt{2}+\sqrt{2}}{2}\right), & s \in [\frac{\pi r+6\pi}{4}, \frac{6\pi+2\pi r}{4}] \end{cases}$$

where $r = \pi\sqrt{2}/(2-\sqrt{2})$.

My wife Melinda, who has put up with far more than she should have waiting for me to graduate.

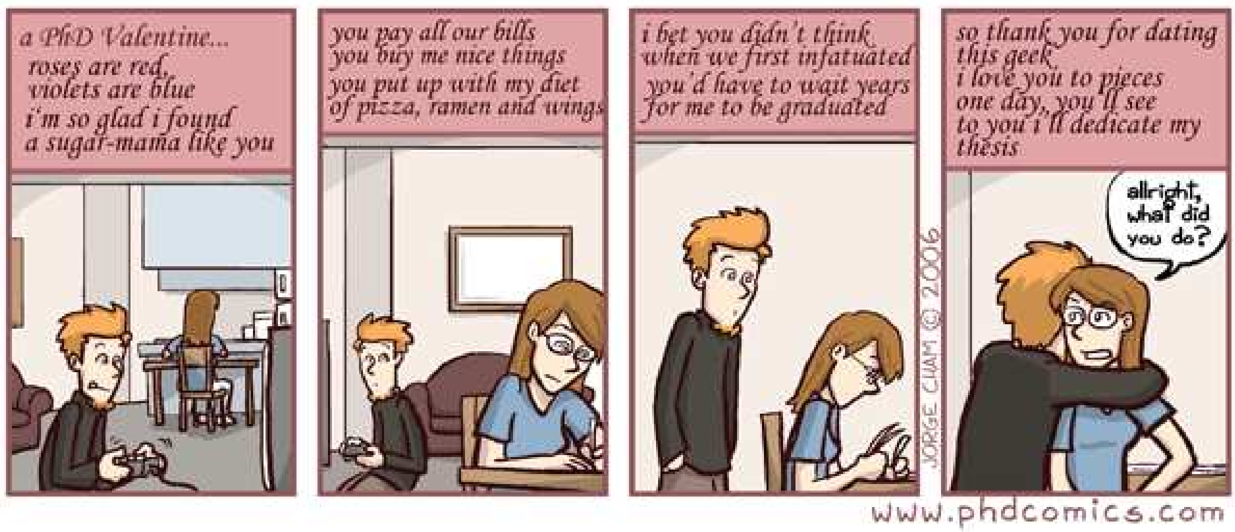

Figure 2: ”Piled Higher and Deeper” by Jorge Cham. www.phdcomics.com

Without my parents this would not have been possible—indeed, without them I (likely) wouldn’t be here. My mother has given me much of her time recently so that I can focus my attention on completing this document. Mom, I promise, at our next place you won’t have to sleep on the floor. To my father, who has been kind enough to proofread, I dedicate the fact that this dissertation is far more analytical than it is topological. I also offer dedication to my sister Lisa, her husband Will, and their son Austin. Thanks for helping hold ol’ Uncle Lou’s chin up. To my beloved cats Yoda, Luke, Leia, and Gremlin I dedicate this dedication. Last, but not least, I dedicate this dissertation to my second child whom I have not yet met.

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A good deal of who I am as a mathematician is a result of my experience at Georgia Tech. Dr. Ghrist, thanks for giving me such a brilliant introduction to Topology. Dr. (Killer) Cain, I greatly appreciate your candor and advice on mathematics and life in general. Dr. Heil, thank you (Master) for all the help you gave and for showing me the beautiful side of analysis. There were many fellow students that I remember fondly, but those that stick out the most are those with whom I took many of my classes. In addition to classmates these folks became good friends. You know you're a geek when you come into the math department on a Saturday to play Dungeons & Dragons. Sean, Todd, Sarah, and Bryan, thanks for the help and companionship. I miss you all.

From my humble beginnings as a mathematician, I owe a great deal to Dr. Faucette, Dr. Gordon, and Dr. McCoy. Thanks for laying such a sturdy foundation. In addition, I would like to acknowledge Jack Miller, whose two calculus classes convinced me to become a mathematician.

To my fellow members of the defunct Round Ear Spock, Fester Venereal, Romeo Six-pack, and Lumpy, and their wives Kim, Brooke, and Kelly...I don't know what to say. Thanks? Seriously, despite some of your attempts to appear the contrary (especially you, Mr. Venereal), you all are some of the nicest and kindest people I have ever met. I reckon when I'm old(er) and (more) grey, I'll look back on the R.E.S. days and our time in Carrollton as the best part of my youth. Last, but not least, I thank Rob Shetler and his wife Ande. Of all my friends, I've known you two the longest and think of you as family. Thanks for all the time. I love you both.

Table of Contents

Chapter 1

Required Background

1.1 The Class of Curves of Interest

This section contains a brief review of general background material and important lemmata and propositions required in the main body of the paper. The importance of this material may only become clear when read in the context of the later chapters. This section also includes motivation and examples that help build intuition on the subject of the main result.

1.1.1 Semicontinuity and Envelopes

It is a well known fact that a continuous function defined on a compact set achieves both its maximum and minimum. Semicontinuity, a condition weaker than continuity, forces a function to achieve either its maximum or its minimum on a compact domain, but not necessarily both.

Definition 1. *An extended real-valued function f is called **lower semicontinuous** at the point y if $f(y) \neq -\infty$ and*

$$f(y) \leq \sup_{\varepsilon>0} \inf_{0<|x-y|<\varepsilon} f(x).$$

*The function f is **upper semicontinuous** at the point y if $f(y) \neq \infty$ and*

$$f(y) \geq \inf_{\varepsilon>0} \sup_{0<|x-y|<\varepsilon} f(x).$$

A function f is upper (lower) semicontinuous on a set A provided it is upper (lower) semicontinuous at all points $y \in A$.

An equivalent definition for upper semicontinuity, which is often more useful, is the following:

Definition 2. *The function $f\colon A \longrightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ is **upper semicontinuous** at x_0 if and only if for each $\alpha > 0$ there exists a $\beta > 0$, so that $f(x) < f(x_0) + \alpha$ whenever $x_0 \in A$ and $|x - x_0| < \beta$.*

The upper (or lower) envelope of a function is a closely related function that is upper (or lower) semicontinuous.

Definition 3. *([16]) Let f be a real-valued function defined on $[a, b]$. We define the **lower envelope** $\underline{f}$ of f to be the function $\underline{f}$ defined by*

$$\underline{f}(x) = \sup_{\varepsilon>0} \inf_{|x-y|<\varepsilon} f(y), \text{ for all } x \in [a, b]$$

*and the **upper envelope** $\overline{f}$ by*

$$\overline{f}(x) = \inf_{\varepsilon>0} \sup_{|x-y|<\varepsilon} f(y) \text{ for all } x \in [a, b].$$

Lemma 4. *Let f be an upper (lower) semicontinuous function. If f is bounded above (below) then f achieves its maximum (minimum) on any compact set.*

Proof. Let K be any compact set and let $B = \sup_{x\in K} f(x)$. There exists a sequence $\{x_i\} \subset K$ so that $f(x_i) \to B$ as $i \to \infty$. Since K is compact, the sequence $\{x_i\}$ contains an accumulation point x and so there exists a subsequence $\{x_{i_j}\} \subset \{x_i\}$ so that $x_{i_j} \to x$ as $j \to \infty$. Since $\{x_{i_j}\}$ is an infinite subsequence of $\{x_i\}$ we know $f(x_{i_j}) \to B$ as $j \to \infty$. Now consider the value of $f(x)$. Since f is upper semicontinuous

$$f(x) \geq \inf_{\varepsilon>0} \sup_{0<|x-y|<\varepsilon} f(y).$$

But in any ε neighborhood of x there is a sequence of values of f converging to B. Thus, for any $\varepsilon > 0$, $\sup_{0<|x-y|<\varepsilon} f(y) = B$. Thus, $\inf_{\varepsilon>0} \sup_{0<|x-y|<\varepsilon} f(y) = B$ and we have shown $f(x) \geq B$. Since $B = \sup_{x\in K} f(x)$, this completes the proof. The proof of the fact that a lower semicontinuous function achieves its minimum on any compact set is similar. $\square$

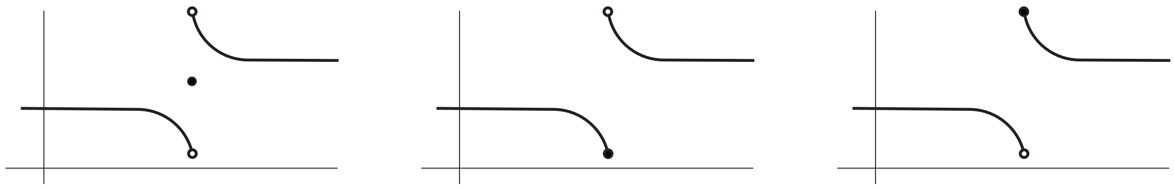

Figure 1.1: The left image is the graph of a function that is defined everywhere, but fails to reach a maximum or minimum. The lower envelope, shown in the center image, is lower semicontinuous and reaches its minimum, and the upper envelope, shown on the right, is upper semicontinuous and reaches its maximum.

We now state and prove the expected relationship between envelopes and semicontinuity.

Lemma 5. *Suppose that f is any real valued function defined on a closed interval $[a, b]$, $\overline{f}$ is the upper envelope of f, and $\underline{f}$ is the lower envelope of f.*

(1) If f is bounded, then $\overline{f}$ is upper semicontinuous and $\underline{f}$ is lower semicontinuous.

(2) If f is bounded, then $\overline{f}$ achieves its maximum at a point in [a,b] and $\underline{f}$ achieves its minimum at a point in $[a, b]$.

(3) For each $x \in [a, b]$ $\underline{f}(x) \leq f(x) \leq \overline{f}(x)$.

Proof. The proof of (1) is immediate from the definitions and the proof of (2) follows from (1) and Lemma 4 on the preceding page.

To see (3), for fixed $x \in [a, b]$, let $\{\varepsilon_i\}$ be an infinite sequence of real numbers with the property that $\varepsilon_i \to 0$ as $i \to \infty$ and

$$\lim_{i\to\infty} \inf_{|x-y|<\varepsilon_i} f(y) = \sup_{\varepsilon>0} \inf_{|x-y|<\varepsilon} f(y).$$

Then for each value of ε_i in the sequence

$$\inf_{|x-y|<\varepsilon_i} f(y) \leq f(x).$$

It follows that $\underline{f}(x) \leq f(x)$ for all values of $x \in [a, b]$. A similar argument shows $f(x) \leq \overline{f}(x)$. $\square$

1.1.2 Rectifiable Curves and Functions of Bounded Variation

We begin with some definitions and notation related to rectifiable curves.

Definition 6. *A **partition** of $[a, b]$ is any finite sequence of points $\{a = x_0, x_1, \ldots, x_n = b\}$ so that $x_i \leq x_j$ whenever $i < j$.*

Definition 7. *Let $\gamma \colon [a, b] \longrightarrow X$ be a curve in the metric space X. To each partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ there is an associated polygon p with vertex set $\{\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)\}$. The set of all such polygons will be denoted $\mathrm{Pol}(\gamma)$ and the number of elements in P will be denoted $|p|$.*

Definition 8. *Let X be a metric space with metric $d(\cdot, \cdot; X)$, and let $\gamma \colon [a, b] \longrightarrow X$ be a curve in X. We define*

$$\mathrm{Len}_X(p) \colon = \sum_{i=1}^{|p|-1} d(\gamma(x_{i-1}), \gamma(x_i); X),$$

*and say γ is X-**rectifiable** provided,*

$$\mathrm{Len}_X(\gamma) \colon = \sup_{p \in \mathrm{Pol}(\gamma)} \mathrm{Len}_X(p) < \infty.$$

If $X = \mathbb{R}^n$ we will write

$$\mathrm{Len}(p) \colon = \mathrm{Len}_{\mathbb{R}^n}(p) = \sum_{i=1}^{|p|-1} d(\gamma(x_{i-1}), \gamma(x_i); \mathbb{R}^n),$$

*and say γ is **rectifiable** provided,*

$$\mathrm{Len}(\gamma) \colon = \sup_{p \in \mathrm{Pol}(\gamma)} \mathrm{Len}(p) < \infty.$$

All of the curves that are used throughout this document will be rectifiable. Since we will often use the length of the curves in our arguments, some notation is appropriate.

Definition 9. *Let X be a metric space and let $\gamma\colon [a,b] \longrightarrow X$ be any X-rectifiable curve in X. If $[s,t] \subset [a,b]$ then the **arclength** between $\gamma(s)$ and $\gamma(t)$ is $\operatorname{Len}_X(\gamma([s,t]))$, the length of γ restricted to the set $[s,t]$, and this arclength will be denoted $d(s,t;\gamma)$. If γ is a closed curve then,*

$$d(s,t;\gamma) := \min\left\{\operatorname{Len}_X(\gamma([s,t])), Len_X(\gamma) - \operatorname{Len}_X(\gamma([s,t]))\right\}.$$

We will frequently make use of the fact that the length of a polygon approximation can only increase if more vertices are added. Since this statement only requires the triangle inequality, it may be stated in the general form as follows.

Lemma 10. *Let X be a metric space and let $\gamma\colon [a,b] \longrightarrow X$ be any X-rectifiable curve. Let $\{p_i\}$ and $\{q_i\}$ be two sequences of polygons inscribed in γ so that $\operatorname{Len}_X(p_i) \to \operatorname{Len}_X(\gamma)$ and so that the vertex set of p_i is a subset of the vertex set of q_i for each i. Then $\operatorname{Len}_X(q_i) \geq \operatorname{Len}_X(p_i)$ for all i and so $\operatorname{Len}_X(q_i) \to \operatorname{Len}_X(\gamma)$ as $i \to \infty$.*

Proof. Let $x_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$ and $y_i = \{y_{i_0}, y_{i_1}, \ldots, y_{i_n}\}$ denote the vertex sets of p_i and q_i respectively. Since $x_i \subset y_i$ it follows from the triangle inequality that

$$\operatorname{Len}_X(p_i) = \sum_{j=0}^{k-1} d(x_{i_j}, x_{i_{j+1}}; X) \leq \sum_{j=0}^{n-1} d(y_{i_j}, y_{i_{j+1}}; X) = \operatorname{Len}_X(q_i).$$

Finally, since $\operatorname{Len}_X(p_i) \leq \operatorname{Len}_X(q_i)$ for all i and since

$$\lim_{i\to\infty} \operatorname{Len}_X(p_i) = \operatorname{Len}_X(\gamma) = \sup_{r \in \operatorname{Pol}(\gamma)} \operatorname{Len}_X(r),$$

it follows that $\operatorname{Len}_X(q_i) \to \operatorname{Len}_X(\gamma)$. □

Our first application of Lemma 10 is found in the proof of the following Lemma 11 concerning the change in length of a curve that has been radially scaled.

Lemma 11. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$, be a rectifiable curve and let C be any positive constant. Let $C\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$ be the curve defined by*

$$(C\gamma)(s) = C\gamma(s).$$

Then $\operatorname{Len}(C\gamma) = C\operatorname{Len}(\gamma)$.

Proof. Let $\mathcal{A}$ denote the arc of $C\gamma$ from $C\gamma(s)$ to $C\gamma(t)$, and let α be the corresponding arc of γ from $\gamma(s)$ to $\gamma(t)$. The length of $\mathcal{A}$ is given by the supremal length of all polygonal arcs inscribed in $\mathcal{A}$ with endpoints at $C\gamma(s)$ and $C\gamma(t)$. Let p be any polygonal arc inscribed in $\mathcal{A}$ with vertices $(C\gamma)(x_0) = (C\gamma)(s), (C\gamma)(x_1), \ldots, (C\gamma)(x_{n-1}), (C\gamma)(x_n) = (C\gamma)(t)$. Then, the length of p is exactly

$$\begin{aligned}\sum_{i=1}^{n} \|(C\gamma)(x_i) - (C\gamma)(x_{i-1})\| &= \sum_{i=1}^{n} \|C\gamma(x_i) - C\gamma(x_{i-1})\| \\ &= \sum_{i=1}^{n} C\|\gamma(x_i) - \gamma(x_{i-1})\|.\end{aligned}$$

Thus there is a one-to-one correspondence between the set of polygons inscribed in α and the set of polygons inscribed in $\mathcal{A}$ given by $p \leftrightarrow Cp$. Let $\{p_i\}$ be any sequence of polygons inscribed in α whose lengths converge to the length of α. Then the sequence Cp_i is a sequence approaching the supremal length of curves inscribed in $\mathcal{A}$. Since $\mathrm{Len}(Cp_i) = C\,\mathrm{Len}(p_i)$ for all i, we have $C\,\mathrm{Len}(\gamma) = \mathrm{Len}(C\gamma)$ as desired.

□

The variation of a function is similar to the length of a curve, and functions of bounded variation are closely related to rectifiable curves. We could work only with the set of rectifiable curves, but there are measure theoretic properties of this class more naturally expressed in the language of functions.

Definition 12. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function. Let P denote the set of all partitions of $[a,b]$. Then the* ***variation*** *of f is given by*

$$\sup_{p\in P} \sum_{i=1}^{|p|-1} |f(p_i) - f(p_{i-1})|$$

where $|p|$ denotes the number of elements in the partition p.

We say f is a ***function of bounded variation*** *whenever the variation of f is bounded. Moreover, we say that a curve $g\colon [a,b] \longrightarrow \mathbb{R}^n$ is of bounded variation provided each of its coordinate functions is of bounded variation.*

The difference between a rectifiable curve and a curve of bounded variation is really only a matter of perspective. It is a theorem in [20] (c.f. [21]) that any continuous parametric curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$, parametrized by $\gamma(s) = (x(s), y(s), z(s))$, is rectifiable if and only if the three coordinate functions $x(s)$, $y(s)$, and $z(s)$ are of bounded variation.

Any rectifiable curve $\gamma\colon [a,b] \to \mathbb{R}^3$ can be parametrized by arclength. To parametrize γ using the arclength parameter s means that given any two points $s_0, s_1 \in [a,b]$, $\mathrm{Len}(\gamma([s_0,s_1])) = |s_1 - s_0|$. We will be assuming, unless otherwise specified, that all of our curves are rectifiable.

1.1.3 Measure Theory

This section provides measure theoretic lemmata required later in the proof of the main theorem. We begin with the definition of a class of measures.

Definition 13. *([17]) A **Radon measure** μ is a Borel measure on $X \subset \mathbb{R}^n$ that satisfies the following properties:*

(1) For all compact sets $K \subset X$, $\mu(K) < \infty$.

(2) For all Borel sets $E \subset X$, $\mu(E) = \inf\{\mu(U) : E \subset U$ and U is open$\}$.

(3) For all open sets $U \subset X$, $\mu(U) = \sup\{\mu(K) : K \subset U$ and K is compact$\}$.

Radon measures are of interest due to their relationship to functions of bounded variation. It is noted in [13] that the distributional derivative of a function of bounded variation is a Radon measure. We will use a Radon measure to describe the curvature of curves whose tangent curve is not C^1.

We now construct the Lebesgue measure, following [17].

Definition 14. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. The function f is **right (left) continuous** at a point $a \in \mathbb{R}$ provided $\lim_{x\to a^+} f(x) = f(a)$ ($\lim_{x\to a^-} f(x) = f(a)$). We say f is right (left) continuous provided it is right (left) continuous for all $a \in \mathbb{R}$.*

Theorem 15. *([17]) If $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ is any increasing right continuous function, then there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all a, b.*

Definition 16. *To say a measure μ is **complete** means that if $\mu(B) = 0$, then $\mu(A) = 0$ for all $A \subset B$.*

Folland notes that Theorem 15 actually provides a measure μ_F, called the Lebesgue-Stieltjes measure associated to F, which is a complete measure whose domain includes all of the Borel sets of $\mathbb{R}$. Further if we define $\mathcal{M}_{\mu_F}$ to be the domain of μ_F then we also have the following useful theorem which asserts that the measure μ_F is a Radon measure.

Theorem 17. *([17]) If $E \subset \mathcal{M}_{\mu_F}$, then*

$$\begin{aligned} \mu_F(E) &= \inf\{\mu_F(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \sup\{\mu_F(K) : K \subset E \text{ and } K \text{ is compact}\}. \end{aligned}$$

Given a σ-finite measure μ and a complex measure λ, there is a decomposition of λ as a sum of two different measures. One of these measures vanishes where μ vanishes while the other is nonzero on sets with μ measure zero, a process not unlike breaking up λ into orthogonal components. Before stating the theorem, let us describe all the elements involved.

Definition 18. *([13]) The measure ν is **absolutely continuous** with respect to μ, written $\nu \ll \mu$, provided $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathbb{R}^n$.*

Definition 19. *([13]) The measures ν and μ are **mutually singular**, written $\nu \perp \mu$, if there exists a Borel subset $B \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus B) = \nu(B) = 0$.*

Definition 20. *([18]) Let $\mathcal{M}$ be a σ-algebra in a set X. Call a countable collection $\{E_i\}$ of members of $\mathcal{M}$ a set theoretic partition of E if $E_i \cap E_j = \emptyset$ whenever $i \neq j$, and if $E = \bigcup E_i$. A **complex measure** μ on $\mathcal{M}$ is then a complex function on $\mathcal{M}$ such that*

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \; (E \in \mathcal{M})$$

for every partition $\{E_i\}$ of E.

We are now prepared to state the decomposition theorem.

Theorem 21. *([18]) **(Lebesgue-Radon-Nikodym)** Let μ be a positive σ-finite measure on a σ-algebra $\mathcal{M}$ in a set X, and let λ be a complex measure on $\mathcal{M}$.*

(1) There is then a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that

$$\lambda = \lambda_a + \lambda_s,\ \lambda_a \ll \mu,\ \lambda_s \perp \mu.$$

If λ is positive and finite, then so are λ_a and λ_s.

(2) There is a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

for every set $E \in \mathcal{M}$.

For an arbitrary Borel measure μ defined on $\mathbb{R}^n$ there need not be any correlation between the diameter of a set and its measure. But if the measure of every single point is zero, then it can be shown that sets of small diameter have small measure. We will only need this lemma in $\mathbb{R}^1$.

Definition 22. *An **atomic measure** is a Borel measure α_x on X so that if $A \subset X$ is any measurable set, then $\alpha_x(A) = \alpha_x(\{x\}) > 0$ if $x \in A$ and $\alpha_x(A) = 0$ if $x \notin A$. We say that the point x is an **atom** of a measure μ if and only if $\mu(\{x\}) > 0$.*

Lemma 23. *Suppose that μ is a Radon measure defined on $\mathbb{R}$ so that $\mu(\{p\}) = 0$ for each $p \in \mathbb{R}$. Then given any finite interval (a, b), we can find a value $L \in \mathbb{R}$ so that the measure of every subinterval (c, d) of (a, b) with length less than L satisfies the inequality $\mu((c, d)) \leq 2\mu((a, b))/3$.*

Proof. We will proceed by the method of contradiction. We assume that for all $L > 0$ there exists an interval $(c, d) \subset (a, b)$ with $d - c < L$ so that $\mu((c, d)) > 2\mu((a, b))/3$. For each positive integer n, let $L(n) = (b - a)/2^n$, and let s_n be an open interval with length less than $L(n)$ so that

$s_n \subset (a,b)$ and $\mu(s_n) > 2\mu((a,b))/3$. Notice that if we define S_n to be the closure of s_n, then $\mu(S_n) = \mu(s_n)$ since μ contains no atoms. Let $C_n = \{C_{n_i}\}$ be the closed cover of $[a,b]$ consisting of the sets $C_{n_i} = [a + (i-1)L(n), a + iL(n)]$ for $i = 1, 2, \ldots, 2^n$. So C_1 consists of two sets of equal length $C_{1_1} = [a, (a+b)/2]$ and $C_{1_2} = [(a+b)/2, b]$.

Claim 24. *There must be some integer i so that $S_i \subset [a, (a+b)/2]$ or $S_i \subset [(a+b)/2, b]$.*

Proof. If there is no such integer, then it follows that $S_i \cap [a, (a+b)/2] \neq \emptyset$ and $S_i \cap [(a+b)/2, b] \neq \emptyset$ for all i. Therefore, $(a+b)/2 \in S_i$ for all i. Since the length of the S_i's is approaching zero it follows that

$$\left\{ \frac{(a+b)}{2} \right\} = \bigcap_{i=1}^{\infty} S_i.$$

But then since μ is a Radon measure property (2) of Definition 13 on page 7 shows us $\mu(\{(a+b)/2\}) > 2\mu((a,b))/3$, which contradicts the assumption that there are no atoms for μ. $\square$

Therefore, by Claim 24 there exists some S_i so that either $S_i \subset C_{1_1}$ or $S_i \subset C_{1_2}$. Hence, either $\mu(C_{1_1}) > 2\mu((a,b))/3$ or $\mu(C_{1_2}) > 2\mu((a,b))/3$. Define that set to be T_1. The next open cover C_2, covers T_1 with exactly two sets. Using the claim again and the fact that $\mu((a,b)\backslash T_1) < \mu((a,b))/3$, we know that exactly one member of C_2 has μ-measure greater than $2\mu((a,b))/3$. Call it T_2. Continue inductively to generate a sequence of closed intervals $\{T_n\}$ with the following properties:

(1) $\mu(T_n) > 2\mu((a,b))/3$ for all n,

(2) the length of T_n is exactly $(b-a)/2^n$ for all n, and

(3) $T_{n+1} \subset T_n$ for all $n \geq 1$, hence $\cap_{n=1}^{\infty} T_n \neq \emptyset$.

Since the intersection of the T_i's is nonempty and since the length of the T_i's is approaching zero, it follows that the intersection of the T_i's is a single point and that the μ-measure of this point is larger than $2\mu((a,b))/3 \geq 0$ as in Claim 24. This again contradicts the assumption that μ contains no atoms. $\square$

1.1.4 CURVES OF FINITE TOTAL CURVATURE

Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be an embedded rectifiable closed curve parametrized by arclength. Let $s_0, s_1, \ldots, s_n$ be a collection of values so that $s_i \in [a,b]$, $s_i < s_{i+1}$ for all $i = 0, 1, \ldots, n-1$. For the sake of notation, the subscripts will be taken modulo $n+1$. Let E_i denote the line segment from $\gamma(s_i)$ to $\gamma(s_{i+1})$ for $i = 0, 1, \ldots, n$. Define the angle α_i between edges E_i and E_{i-1} to be the angle between the vectors $\gamma(s_{i+1}) - \gamma(s_i)$ and $\gamma(s_i) - \gamma(s_{i-1})$ satisfying $\alpha_i \in [0, \pi]$.

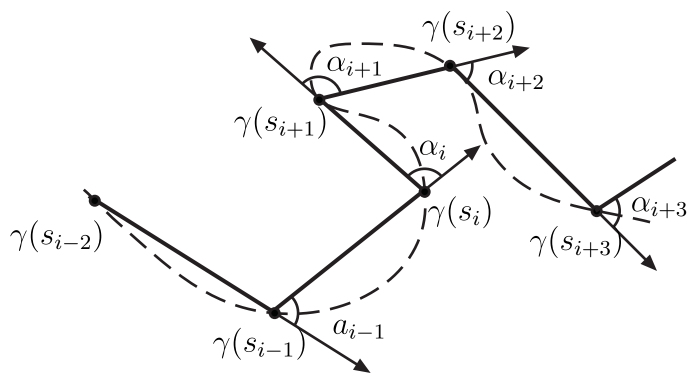

Figure 1.2: Illustration of elements involved in Milnor's definition of curvature.

Definition 25. *The* ***total curvature of the inscribed polygon*** p *given by the* s_i*'s and* E_i*'s as defined by Milnor is* $\kappa(p) = \sum_{i=1}^{n} \alpha_i$*. The* ***total curvature of*** γ*,* ***in the sense of Milnor****, is given by*

$$\kappa([a,b]) = \sup_{p \in \mathrm{Pol}(\gamma)} (\kappa(p)).$$

It is shown in [5] that if γ is a C^2 curve, then Milnor's definition of total curvature agrees with the standard definition $\kappa(\gamma) = \int_0^{\mathrm{Len}(\gamma)} |\gamma''(s)| \,\mathrm{d}s$. For this reason, it is appropriate to denote Milnor's curvature by κ.

We can define the total curvature of any subarc (s_0, s_1) of γ similarly. We will see that extending this measure on intervals to the unique compatible Borel measure gives the Milnor curvature measure $\mathcal{K}$ on $[a,b]$, which is defined so that for any open interval $(s_0, s_1) \subset [a,b]$ we have $\kappa((s_0, s_1)) = \mathcal{K}((s_0, s_1))$.

Definition 26. *We say that* $\gamma \colon [a,b] \longrightarrow \mathbb{R}^3$ *is a curve of* ***finite total curvature*** *provided* γ *is rectifiable, and* $\kappa([a,b]) < \infty$*. The set of all such curves is denoted* FTC.

Another way to interpret the values of the α_i's in Milnor's definition of curvature is to realize that they are the lengths of great circle arcs joining successive tangent vectors on the unit sphere. This gives us another formulation of the total curvature. To make it precise, we need to establish a few facts.

Lemma 27. *If* $\gamma \colon [a,b] \longrightarrow \mathbb{R}^3$ *is a curve with finite total curvature, then* γ *has one-sided tangents everywhere.*

Proof. Recall that the one sided tangent of a curve γ at the point s is given by

$$\gamma'_-(s) = \lim_{t \to s^-} \frac{\gamma(s) - \gamma(t)}{\|\gamma(s) - \gamma(t)\|}.$$

If we let $s \in [a,b]$ and let $\{s_i\}$ be any infinite sequence so that $s - s_i \to 0^+$ monotonically as $i \to \infty$, then a one-sided tangent vector at the point $\gamma(s)$ is given by

$$\gamma'_-(s) = \lim_{i \to \infty} \frac{\gamma(s) - \gamma(s_i)}{\|\gamma(s) - \gamma(s_i)\|}$$

whenever this limit exists. To see that γ has one-sided tangents everywhere, we proceed by contradiction. Without loss of generality, assume that $\gamma(s) = 0$ and that the one-sided tangent at $\gamma(s)$ does not exist. Then, let $\{s_i\}$ be some sequence with $s - s_i \to 0^+$so that the limit

$$\lim_{i \to \infty} \frac{\gamma(s_i)}{\|\gamma(s_i)\|}$$

does not exist. Let $\{\hat{\gamma}(s_i)\}$ denote the sequence $\{\gamma(s_i)/\|\gamma(s_i)\|\}$. Now $\{\hat{\gamma}(s_i)\}$ is an infinite collection of points on S^2, and since S^2 is compact, it follows that there must be at least one accumulation point. Let a_1 be such an accumulation point. If every open neighborhood U of a_1 has the property that all but finitely many elements of $\{\hat{\gamma}(s_i)\}$ are contained in U, then $\lim_{i \to \infty} \hat{\gamma}(s_i) = a_1$, which is a contradiction. So, there must exist an open neighborhood U_1 of a_1 that is disjoint from an infinite subsequence $\{\hat{\gamma}(s_{i_j})\}$ of $\{\hat{\gamma}(s_i)\}$. Again, by compactness of S^2, the sequence $\{\hat{\gamma}(s_{i_j})\}$ must contain at least one accumulation point a_2. In addition, there must exist an open neighborhood U_2 of a_2 that contains infinitely many elements of $\{\hat{\gamma}(s_{i_j})\}$. By restricting to smaller open neighborhoods, we may assume that there exists a constant $C > 0$ so that U_1 and U_2 are open balls centered at a_1 and a_2 respectively and

$$\inf_{x \in U_1, y \in U_2} d(x, y; S^2) \geq C. \tag{1.1}$$

By restricting to a subsequence, we may assume that $\{s_i\}$ is a sequence so that

(1) $\gamma(s_i) \neq \gamma(s_j)$ for all $i \neq j$,

(2) $s - s_i \to 0^+$ monotonically as $i \to \infty$ as before,

(3) $\hat{\gamma}(s_{2k+1}) \in U_1$ for all k, and

(4) $\hat{\gamma}(s_{2k}) \in U_2$ for all k.

We will now construct a polygon inscribed in γ with arbitrarily high curvature, contradicting the assumption that $\gamma \in \mathrm{FTC}$.

Let $v_1 = \gamma(s_1)$. We will now show that there exists some value k so that the direction of the edge with vertices v_1 and $v_2 = \gamma(s_k)$ lies within U_1. The following lemma must first be proven.

Lemma 28. *Let $w \in \mathbb{R}^3 - \{0\}$ and let $\{w_i\} \subset \mathbb{R}^3 - \{0\}$ be a sequence so that $\|w - w_i\| \to 0$ as $i \to \infty$. If $\hat{w} = w/\|w\|$ and $\hat{w}_i = w_i/\|w_i\|$, then*

$$\|\hat{w} - \hat{w}_i\| \to 0 \textit{ as } i \to \infty.$$

Proof. First, define $\|w\| = A$ and $\|w_i\| = A_i$, and let $\varepsilon > 0$. Because $\|w - w_i\| \to 0$, there exists an integer N so that $|A - A_i| \leq \varepsilon A/2$ and $\|w - w_i\| < \varepsilon A/2$ when $i > N$. Then,

$$\|\hat{w} - \hat{w}_i\| = \left\| \frac{wA_i - w_iA}{AA_i} \right\| = \frac{1}{AA_i} \|wA_i - w_iA\|$$

Now observe that

$$\|wA_i - w_iA\| = \|A_i(w - w_i) + w_i(A_i - A)\|$$

$$\leq (A_i\|w - w_i\| + \|w_i\||A_i - A|)$$

$$= (A_i\|w - w_i\| + A_i|A_i - A|)$$

Using the assumptions that $|A - A_i| \leq \varepsilon A/2$ and $\|w - w_i\| < \varepsilon A/2$, we see

$$A_i\|w - w_i\| + A_i|A_i - A| < \left(A_i\left(\frac{\varepsilon A}{2}\right) + A_i\left(\frac{\varepsilon A}{2}\right)\right)$$

$$= \varepsilon AA_i.$$

Thus

$$\|\hat{w} - \hat{w}_i\| < \varepsilon.$$

$\square$

Since the points $\gamma(s_i) \to 0$ as $i \to \infty$, given $\varepsilon > 0$, there exists a positive integer N so that $\|\gamma(s_i)\| < \varepsilon$ whenever $i > N$. Hence, by making ε small enough, it follows from Lemma 28 that the direction of a segment with end points $v_1 = \gamma(s_1)$ and $v_2 = \gamma(s_{2N})$ lies in U_1.

In a similar fashion, since we know $\gamma(\hat{s_{2N}}) \in U_2$ and since $\|\gamma(s_i)\| \to 0$ as $i \to \infty$, we can find a value k large enough so that direction of the edge with endpoints v_2 and $v_3 = \gamma(s_{2k+1})$ lies within U_2. By equation (1.1), the total curvature of these first two edges is at least C. Continuing in this fashion we can construct a polygon inscribed in the arc of γ from $\gamma(s_1)$ to $\gamma(s)$ with vertices $v_1, v_2, \ldots, v_n$ whose total curvature is at least $(n-1)C$.

Letting $n \to \infty$ we have demonstrated a sequence of inscribed polygons whose curvature grows arbitrarily large. Therefore $\gamma \notin \mathrm{FTC}$, and we have reached the desired contradiction. A similar argument shows that the right sided tangent also exists. We conclude that γ has both one-sided tangents at $\gamma(s)$. $\square$

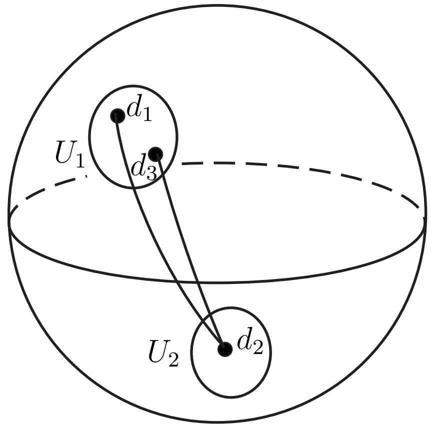

Figure 1.3: Here we see points on $d_1, d_2, d_3 \in S^2$ that are the directions of the first three edges of our construction. The spherical distance between U_1 and U_2 is at least C, hence the length of the great circle arc connecting d_1 to d_2 is at least C, as is the length of the arc connecting d_2 to d_3. Therefore, the total curvature of the polygon constructed so far is at least $2C$.

Although FTC curves are not differentiable everywhere, Lemma 27 allows us to define a function similar to the tangent indicatrix of a differentiable curve.

Definition 29. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3 \in \mathrm{FTC}$ be parametrized by arclength. We define the* ***one-sided tangent indicatrix*** *$T_\gamma^\pm\colon [a,b] \longrightarrow S^2$ by*

$$T_\gamma^\pm(s) = \lim_{s_i \to s^\pm} \frac{\gamma(s_i) - \gamma(s)}{\|\gamma(s_i) - \gamma(s)\|}.$$

We will now show that the length of either one-sided tangent indicatrix is equal to Milnor's total curvature for curves in FTC. We will need one last technical lemma to do this since the length of a space curve is measured by the supremal lengths of inscribed polygons and Milnor's total curvature is measured by the supremal length of curves made up of unions of great circle arcs. We begin with a definition and will proceed to prove a general statement from which the desired result will follow.

Definition 30. *Let Y and Z be metric spaces with $Y \subset Z$. We say the metrics $d(\cdot,\cdot;Y)$ and $d(\cdot,\cdot;Z)$ are **globally compatible** with constant C provided for all $y_1, y_2 \in Y$*

$$\frac{1}{C} \leq \frac{d(y_1, y_2; Y)}{d(y_1, y_2; Z)} \leq C.$$

Lemma 31. *Let Y and Z be metric spaces with $Y \subset Z$ so that their metrics are globally compatible with constant C. If $\gamma\colon [a,b] \longrightarrow Y$ is a curve that is either Y-rectifiable or Z-rectifiable, then it is both Y-rectifiable and Z-rectifiable and*

$$\frac{1}{C}\,\mathrm{Len}_Z(\gamma) \leq \mathrm{Len}_Y(\gamma) \leq C\,\mathrm{Len}_Z(\gamma).$$

Proof. Let $Y \subset Z$ be globally compatible metric spaces. Let $\gamma\colon [a,b] \longrightarrow Y$ be a Z-rectifiable curve in Y. Since γ is Z-rectifiable we know that $\mathrm{Len}_Z(\gamma) = \sup_{p\in \mathrm{Pol}(\gamma)} \mathrm{Len}_Z(p)$ and $\mathrm{Len}_Y(\gamma) = \sup_{p\in \mathrm{Pol}(\gamma)} \mathrm{Len}_Y(p)$. Since Y and Z are globally compatible metric spaces with constant C we have for each polygon p with vertices $x_0, x_1, \ldots, x_n$

$$\sum_{i=1}^{n} \frac{1}{C} d(x_{i-1}, x_i; Z) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i; Y) \leq \sum_{i=1}^{n} C d(x_{i-1}, x_i; Z).$$

Hence,

$$\frac{1}{C}\,\mathrm{Len}_Z(p) \leq \mathrm{Len}_Y(p) \leq C\,\mathrm{Len}_Z(p).$$

Since the above equation holds for all $p \in \mathrm{Pol}(\gamma)$ it follows that

$$\frac{1}{C}\,\mathrm{Len}_Z(\gamma) \leq \mathrm{Len}_Y(\gamma) \leq C\,\mathrm{Len}_Z(\gamma)$$

$\square$

Corollary 32. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be any curve of finite total curvature $\kappa(\gamma)$. Then, if $T_\gamma^+ : [a,b) \longrightarrow S^2$ and $T_\gamma^- \colon (a,b] \longrightarrow S^2$ denote the one-sided tangent indicatrices of γ, then there exists a constant C so that*

$$\frac{2}{\pi}\,\mathrm{Len}(T_\gamma^-) \leq \kappa(\gamma) \leq \frac{\pi}{2}\,\mathrm{Len}(T_\gamma^-)$$

and

$$\frac{2}{\pi}\,\mathrm{Len}(T_\gamma^+) \leq \kappa(\gamma) \leq \frac{\pi}{2}\,\mathrm{Len}(T_\gamma^+).$$

Proof. The total curvature $\kappa(\gamma)$ of γ is the S^2-length of $T_\gamma^\pm$ while $\mathrm{Len}(T_\gamma^\pm)$ is the $\mathbb{R}^3$-length. By Lemma 31, the inequalities in the statement of the corollary are satisfied provided S^2 and $\mathbb{R}^3$ are globally compatible metric spaces. Thus it suffices to show that given any two points $x_1, x_2 \in S^2$, there exists a constant C so that $d(x_1, x_2; \mathbb{R}^3)/C \leq d(x_1, x_2; S^2) \leq Cd(x_1, x_2; \mathbb{R}^3)$. To see this, let $x_1, x_2 \in S^2$, and consider the plane that contains the great circle arc between them. Rotate the plane so that the straight line segment x_1x_2 is vertical as shown in Figure 1.4.

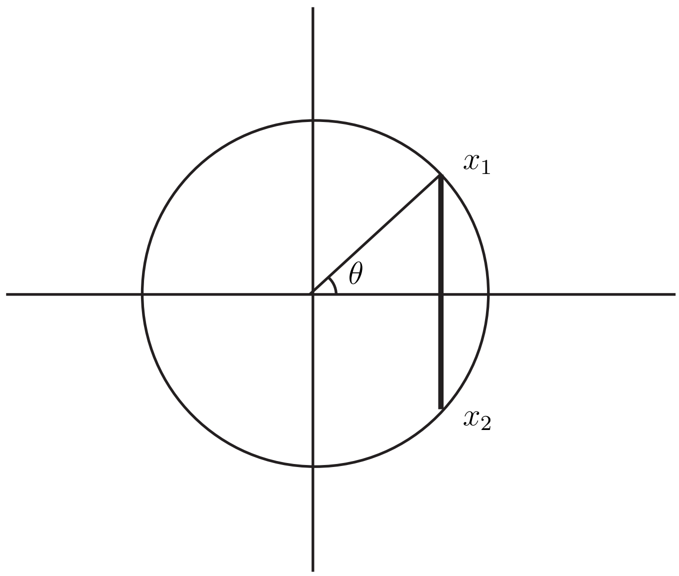

Figure 1.4: Here we observe that $d(x_1, x_2; S^2) = 2\theta$ and $d(x_1, x_2; \mathbb{R}^3) = 2\sin\theta$, where $\theta \in [0, \pi/2]$.

So,

$$\frac{d(x_1, x_2; S^2)}{d(x_1, x_2; \mathbb{R}^3)} = \frac{\theta}{\sin\theta},$$

where $\theta \in [0, \pi/2]$ and thus

$$d(x_1, x_2; S^2) = \frac{\theta}{\sin\theta} d(x_1, x_2; \mathbb{R}^3).$$

Since $\theta \in [0, \pi/2]$ we know that $1 \geq \theta/\sin\theta \leq \pi/2$. Therefore

$$\frac{2}{\pi} d(x_1, x_2; \mathbb{R}^3) \leq d(x_1, x_2; S^2) \leq \frac{\pi}{2} d(x_1, x_2; \mathbb{R}^3)$$

and this completes the proof. $\square$

A property of curves in FTC needed in the next chapter is that the curvature of such a curve is a Radon measure. Recall that the structure theorem for functions of bounded variation in [13] asserts that the derivative of a function of bounded variation is a Radon measure.

Proposition 33. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$. Then $\gamma \in \mathrm{FTC}$ if and only if both of the one-sided tangent indicatrices of γ are functions of bounded variation.*

Proof. First suppose that $\gamma \in \mathrm{FTC}$ and that γ is parametrized by arclength. Then we are to show that the one-sided tangent indicatrices $T_\gamma^-\colon (a,b] \longrightarrow S^2$ and $T_\gamma^+\colon [a,b) \longrightarrow S^2$ are of bounded variation. By [20] a curve's component functions are of bounded variation (and thus the curve is of bounded variation) if and only if the curve is rectifiable. Hence, it suffices to show that T_γ^+ and T_γ^- are $\mathbb{R}^3$-rectifiable, but this is immediate from Corollary 32.

Now, if $T_\gamma^\pm$ are of bounded variation, then their coordinate functions are of bounded variation and hence they are $\mathbb{R}^3$-rectifiable. The result follows from Lemma 31. $\square$

We are now prepared to construct the curvature measure. We define the following function to be used in conjunction with Theorem 15 on page 8.

Definition 34. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve of finite total curvature parametrized by arclength. If $\gamma(a) = \gamma(b)$, we define the function $F_\mathcal{K}\colon \mathbb{R} \longrightarrow \mathbb{R}$ by*

$$F_\mathcal{K}(x) := \begin{cases} 0 & x \leq a \\ \mathrm{Len}_{S^2}(T_\gamma^-((a,x])) + d(T_\gamma^+(x), T_\gamma^-(x); S^2) & x \in (a,b] \\ \mathrm{Len}_{S^2}(T_\gamma^-([a,b])) + d(T_\gamma^+(b), T_\gamma^-(b); S^2) & x > b. \end{cases}$$

If γ is an open curve, then

$$F_\mathcal{K}(x) := \begin{cases} 0 & x \leq a \\ \mathrm{Len}_{S^2}(T_\gamma^-((a,x])) + d(T_\gamma^+(x), T_\gamma^-(x); S^2) & x \in (a,b) \\ \mathrm{Len}_{S^2}(T_\gamma^-([a,b))) & x \geq b. \end{cases}$$

Lemma 35. *The function $F_\mathcal{K}$ is both increasing and right continuous.*

Proof. Certainly $F_{\mathcal{K}}$ is increasing since it measures the S^2-length of arcs of T_γ^-. Let $y \in \mathbb{R}$ and let $\{y_n\}$ be any sequence so that $y_n \to y^+$. If $y \leq a$, then there exists a constant M so that $F_{\mathcal{K}}(y_n) = 0 = F_{\mathcal{K}}(y)$ for all $n > M$. A similar argument shows that $F_{\mathcal{K}}$ is right continuous for all $y \geq b$. Now assume that $y \in (a, b)$ and again let $\{y_n\}$ be any sequence so that $y_n \to y^+$. All but finitely many terms in the sequence $\{y_n\}$ must lie within the interval (a, b), therefore assume all elements of the sequence $\{y_n\}$ lie within (a, b). Then, for each n

$$F_{\mathcal{K}}(y_n) = \mathrm{Len}_{S^2}(T_\gamma^-((a, y_n])) + d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2).$$

and we are to show

$$\left|\mathrm{Len}_{S^2}(T_\gamma^-((a, y_n])) + d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2) - \mathrm{Len}_{S^2}(T_\gamma^-((a, y])) - d(T_\gamma^+(y), T_\gamma^-(y); S^2)\right| \to 0.$$

First note that if T_γ^- is continuous at y then

$$\mathrm{Len}_{S^2}(T_\gamma^-((a, y_n])) = \mathrm{Len}_{S^2}(T_\gamma^-((a, y])) + \mathrm{Len}_{S^2}(T_\gamma^-((y, y_n])).$$

If T_γ^- is not continuous at y, then there is a jump of S^2-distance $d(T_\gamma^+(y), T_\gamma^-(y); S^2)$ between the points $T_\gamma^-(y)$ and $T_\gamma^+(y)$ and so

$$\mathrm{Len}_{S^2}(T_\gamma^-((a, y_n])) = \mathrm{Len}_{S^2}(T_\gamma^-((a, y])) + \mathrm{Len}_{S^2}(T_\gamma^-((y, y_n])) + d(T_\gamma^+(y), T_\gamma^-(y); S^2).$$

The above equation holds even when T_γ^- is continuous since, in that case, $d(T_\gamma^+(y), T_\gamma^-(y); S^2) = 0$. Therefore

$$\begin{aligned} &\left|\mathrm{Len}_{S^2}(T_\gamma^-((a, y_n]) + d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2) - \mathrm{Len}_{S^2}(T_\gamma^-((a, y]) - d(T_\gamma^+(y), T_\gamma^-(y); S^2)\right| \\ &= \mathrm{Len}_{S^2}(T_\gamma^-((y, y_n])) + d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2). \end{aligned}$$

Since $\gamma \in \mathrm{FTC}$ the contribution to the total curvature at corners must be finite. Therefore,

$$\sum_{n=0}^{\infty} d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2) < \infty$$

and so

$$d(T_\gamma^+(y_n), T_\gamma^-(y_n); S^2) \to 0.$$

Recall that since $\gamma \in \mathrm{FTC}$, Proposition 33 asserts that T_γ^- is of bounded variation, hence it is rectifiable. Therefore, $\mathrm{Len}_{S^2}(T_\gamma^-((y, y_n])) \to 0$ since $|y - y_n| \to 0$. $\square$

Theorem 36. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve of finite total curvature parametrized by arclength. Then there is a unique Radon measure $\mathcal{K}$ on $\mathbb{R}$ such that*

$$\mathcal{K}_F((c,d]) = F_{\mathcal{K}}(d) - F_{\mathcal{K}}(c)$$

for all c, d.

Proof. This follows immediately from the definition of $F_{\mathcal{K}}$, Lemma 35 on page 18, Theorem 15 on page 8, and Theorem 17 on page 8. $\square$

It is interesting to note that the measure $\mathcal{K}$ is not absolutely continuous with respect to arclength. Indeed, $\mathcal{K}$ contains an atom whenever the left and right tangent vectors on γ are not equal. This occurs at corner or cusp points along γ. Further, the measure $\mathcal{K}((a,b))$ of any interval (a,b) is equal to the total curvature of that interval in the sense of Milnor $\kappa((a,b))$.

1.2 Distortion

We begin with a definition from Gromov's 1981 paper "Filling Riemannian Maniflods."

Definition 37. *Let V and W be metric spaces with metrics $d(\cdot,\cdot;V)$ and $d(\cdot,\cdot;W)$ respectively and let $f\colon V \longrightarrow W$ be a map. Then, the **dilation** of f is a Lipschitz constant for f given by*

$$\mathit{Dil}\ f = \sup_{v_1,v_2 \in V} \frac{d(f(v_1), f(v_2); W)}{d(v_1, v_2; V)}.$$

*In the event that f is a homeomorphism of V onto a submanifold W of $\mathbb{R}^n$ where W is equipped with the Euclidean metric (i.e., $d(\cdot,\cdot;W) = d(\cdot,\cdot;\mathbb{R}^n)$) we can define the **distortion** of f to be the* *product*

$$[\textit{Dil } f]\left[\textit{Dil } f^{-1}\right] = \left[\sup_{v_1,v_2\in V} \frac{d(f(v_1),f(v_2);\mathbb{R}^n)}{d(v_1,v_2;V)}\right]\left[\sup_{w_1,w_2\in W} \frac{d(f^{-1}(w_1),f^{-1}(w_2);V)}{d(w_1,w_2;\mathbb{R}^n)}\right].$$

Gromov notices that all closed curves in $\mathbb{R}^n$ have distortion at least $\pi/2$, with equality only for the round circle (see Proposition 45 on page 33). He also asks a question which is still open: "Does *every* isotopy class of knots in $\mathbb{R}^3$ have a representative V in $\mathbb{R}^3$ with distortion < 100? Is it so for all torus knots $T_{p,q}$ for $p, q \to \infty$?"

It seems likely that the answer to Gromov's first question is "no." In order for a curve to have high distortion, there must exist points on the curve that are close together in space and yet far apart along the knot. If an entire knot type is to have high distortion, it must be the case that *any* embedding of the knot contains a pair of such points. A candidate for a knot type of high distortion is the $(n, n-1)$-torus knot for some large value of n. Indeed, as the value of n increases the distortions of the curves given by the parametrizations in Figure 1.5 approach infinity.

Since we are only concerned with the distortion of knots, we will start by defining distortion in a more specific form. We assume the manifold V is the circle S^1, and that the embedding takes place in $\mathbb{R}^3$. The homeomorphism, then, is of the form $\gamma\colon S^1 \longrightarrow W$, where $W \subset \mathbb{R}^3$ is equipped with the Euclidean metric, and in this case the distortion can be simplified somewhat.

Lemma 38. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a simple closed curve parametrized by arclength. Then Dil $\gamma = 1$ and so the distortion of γ is Dil γ^{-1}.*

Proof. Since γ is rectifiable,

$$\operatorname{Len}(\gamma) = \sup_{p\in \operatorname{Pol}(\gamma)} \operatorname{Len}(p) < \infty.$$

Let $\varepsilon > 0$, and suppose that $\{p_i\}$ is a sequence of polygons inscribed in γ so that $\lim_{i\to\infty} \operatorname{Len}(p_i) = \operatorname{Len}(\gamma)$. Each p_i has a sequence of vertices $\{\gamma(x_{i_j})\}$ where $j = 1, \ldots, n_i$. In addition, since the polygons are inscribed, to each edge e_{i_j} with endpoints $\gamma(x_{i_j})$ and $\gamma(x_{i_{j+1}})$, there is an associated arc $a_{i_j} := \gamma([x_{i_j}, x_{i_{j+1}}])$ of γ, whose endpoints are equal to the endpoints of e_{i_j}. Since $\lim_{i\to\infty} p_i$

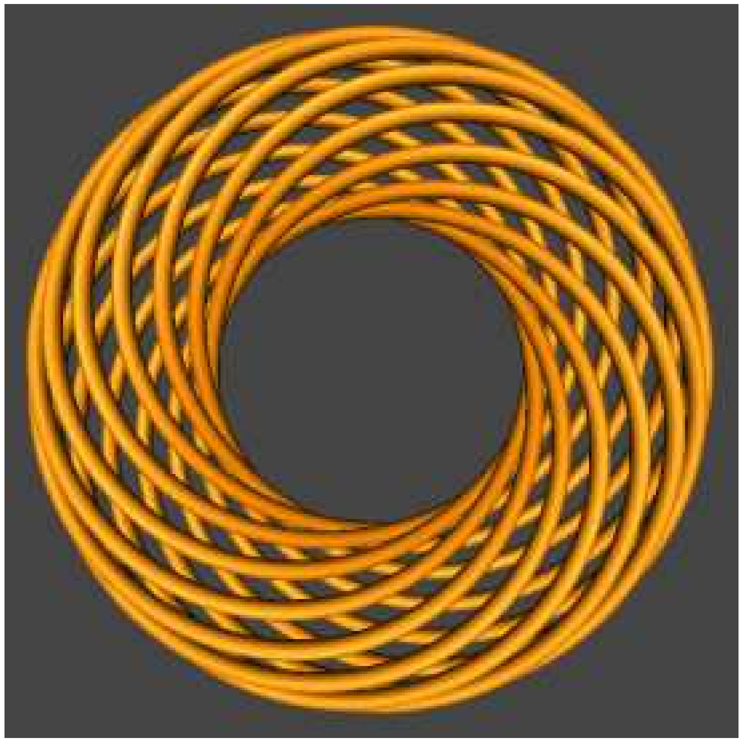

Figure 1.5: Here we see an $(n, n-1)$-torus knot (where $n = 14$), which has been parametrized by $(3 + \cos(2\pi(n-1)t)\cos(2\pi nt), 3 + \cos(2\pi(n-1)t)\sin(2\pi nt), \sin(2\pi(n-1)t))$ where $t \in [0, 1]$.

is converging to the length of γ, we know that there is a positive integer N so that $\mathrm{Len}(p_i) > (1-\varepsilon)\,\mathrm{Len}(\gamma)$ for all $i > N$. This condition can be rewritten as

$$\sum_{j=1}^{n_i} d(\gamma(x_{i_j}), \gamma(x_{i_{j+1}}); \mathbb{R}^3) > \sum_{j=1}^{n_i}(1-\varepsilon)d(\gamma(x_{i_j}), \gamma(x_{i_{j+1}}); \gamma)$$

for all $i > N$. This implies that there is at least one edge e_{i_k} of p_i so that

$$d(\gamma(x_{i_k}), \gamma(x_{i_{k+1}}); \mathbb{R}^3) \geq (1-\varepsilon)d(\gamma(x_{i_k}), \gamma(x_{i_{k+1}}); \gamma) = \varepsilon)d(\gamma(x_{i_k}), \gamma(x_{i_{k+1}}); S^1).$$

Hence

$$\text{Dil}\,\gamma \geq \frac{d(\gamma(x_{i_k}), \gamma(x_{i_{k+1}}); \mathbb{R}^3)}{d(x_{i_k}, x_{i_{k+1}}; S^1)} \geq 1 - \varepsilon.$$

Letting $\varepsilon \to 0$, we have shown that $\text{Dil}(\gamma) \geq 1$. We now observe that $\text{Dil}(\gamma) \leq 1$. Since γ is parametrized by arclength, we know that the value $d(\gamma(x), \gamma(y), \mathbb{R}^3)$ is at most $|x - y|$ and hence $\text{Dil}(\gamma) \leq 1$. $\square$

Given Lemma 38, the definition of distortion can be written as follows for rectifiable curves:

Definition 39. *Let $\gamma\colon [a, b] \longrightarrow \mathbb{R}^3$ be a rectifiable simple closed curve. The* ***distortion quotient*** *of γ at the point (s, t) is given by*

$$dq_\gamma(s, t) \colon = \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)}.$$

The ***distortion*** *of the curve γ, denoted $\delta(\gamma)$, is then*

$$\delta(\gamma) \colon = \sup_{s \neq t} dq_\gamma(s, t).$$

Finally, if we let $[\gamma]$ denote the knot type of γ, then we can define the distortion of the entire knot type to be

$$\delta([\gamma]) \colon = \inf_{\gamma \in [\gamma]} \delta(\gamma).$$

Notice that the distortion of a straight line is 1. If we replace the Euclidean 3-space in the definition of distortion with another Riemannian manifold, then geodesics will have distortion 1 (at least locally).

1.2.1 Scale Invariance

Gromov's questions about distortion remain unanswered at the time of this writing because minimizing distortion among the set of curves in a knot type is a very tricky endeavor. In fact, it is not currently known that a distortion minimizer exists for any knot type other than the unknot. One technical obstacle to proving the existence of distortion minimizers is the fact that the distortion is scale invariant.

Proposition 40. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a rectifiable curve, and let C be any positive constant. Let $C\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be the curve defined by*

$$(C\gamma)(s) = C\gamma(s).$$

Then $dq_\gamma(s,t) = dq_{C\gamma}(s,t)$ for all points $(s,t) \in [a,b] \times [a,b]$ with $s \neq t$.

Proof. Since $\|Cs - Ct\| = C\|s-t\|$ for all positive C, it follows that the denominator in $dq_{C\gamma}(s,t)$ is equal to the C times the denominator of $dq_\gamma(s,t)$ for all points $(s,t) \in [a,b] \times [a,b]$ with $s \neq t$. By Lemma 11 on page 5, $d(x,y,C\gamma) = Cd(x,y,\gamma)$, so we can conclude $dq_{C\gamma}(s,t) = dq_\gamma(s,t)$ for all points $(s,t) \in [a,b] \times [a,b]$ with $s \neq t$. □

1.2.2 Equal Angle Condition

In the definition of distortion, we take the supremum of dq_γ over all pairs of distinct points on the curve. We can rewrite this supremum for curves by thinking of the space $\gamma \times \gamma$ as a family of sets of pairs of points with arclength distance ℓ and taking the supremum over all values of $\ell \leq \mathrm{Len}(\gamma)/2$ if γ is closed and $\ell \leq \mathrm{Len}(\gamma)$ if γ is open. We have, if γ is closed

$$\delta(\gamma) = \sup_{\ell \leq \mathrm{Len}(\gamma)/2} \left[\sup_{x \in \mathbb{R}/\mathrm{Len}(\gamma)} \frac{\ell}{\|\gamma(x) - \gamma(x+\ell)\|} \right].$$

Similarly, if γ is open we have,

$$\delta(\gamma) = \sup_{\ell \leq \mathrm{Len}(\gamma)} \left[\sup_{x \in \mathbb{R}/\mathrm{Len}(\gamma)} \frac{\ell}{\|\gamma(x) - \gamma(x+\ell)\|} \right].$$

If the curve is smooth enough, we can take derivatives to find conditions on pairs of points which realize the supremum. This gives us the following proposition.

Proposition 41. *Suppose that γ is an embedded closed curve in $\mathbb{R}^3$ parametrized by arclength. Suppose $x \neq y$, γ is differentiable at $\gamma(x)$ and $\gamma(y)$, and $dq_\gamma(x,y) = \delta(\gamma)$. In addition suppose $d(\gamma(x), \gamma(y); \gamma) \neq \mathrm{Len}(\gamma)/2$ if γ is closed and $d(\gamma(x), \gamma(y); \gamma) \neq \mathrm{Len}(\gamma)$ if γ is open. Then,*

$$\langle \gamma(x) - \gamma(y), \gamma'(x) \rangle = \langle \gamma(x) - \gamma(y), \gamma'(y) \rangle, \tag{1.2}$$

and

$$\left\langle \frac{\gamma(x) - \gamma(y)}{\|\gamma(x) - \gamma(y)\|}, \gamma'(y) \right\rangle = -\frac{1}{dq_\gamma(x, y)} = \left\langle \frac{\gamma(x) - \gamma(y)}{\|\gamma(x) - \gamma(y)\|}, \gamma'(x) \right\rangle \tag{1.3}$$

That is, the angle between the chord $\gamma(x)\gamma(y)$ and the tangent vector at $\gamma(x)$ is equal to the angle between the chord $\gamma(x)\gamma(y)$ and the tangent vector at $\gamma(y)$. In addition, the angle between the chord $\gamma(x)\gamma(y)$ and $\gamma'(x)$ (or $\gamma'(y)$) is equal to the negative reciprocal of the distortion quotient evaluated at (x, y).

Proof. First assume that if γ is closed, then $d(\gamma(x), \gamma(y); \gamma) \neq \mathrm{Len}(\gamma)/2$ and if γ is open, then $d(\gamma(x), \gamma(y); \gamma) \neq \mathrm{Len}(\gamma)$. If γ is differentiable at $\gamma(x)$ and $\gamma(x + \ell)$ and $dq_\gamma(x, x + \ell) = \delta(\gamma)$ then the function $\ell/\|\gamma(x) - \gamma(x + \ell)\|$ is differentiable at (x, ℓ) and its partial derivatives must vanish. We now compute these partial derivatives.

$$\frac{\partial}{\partial x} \frac{\ell}{\|\gamma(x) - \gamma(x + \ell)\|} = -\ell \frac{\langle \gamma(x) - \gamma(x + \ell), \gamma'(x) - \gamma'(x + \ell) \rangle}{\|\gamma(x) - \gamma(x + \ell)\|^3}$$

$$\frac{\partial}{\partial \ell} \frac{\ell}{\|\gamma(x) - \gamma(x + \ell)\|} = \frac{\|\gamma(x) - \gamma(x + \ell)\|^2 - \ell \langle \gamma(x) - \gamma(x + \ell), -\gamma'(x + \ell) \rangle}{\|\gamma(x) - \gamma(x + \ell)\|^3}$$

The case when $\|\gamma(x) - \gamma(x+\ell)\| = 0$ can only happen when $\ell = 0$ since the curve is embedded and we assumed $\ell \neq 0$. When the numerators above vanish,

$$\langle \gamma(x) - \gamma(x + \ell), \gamma'(x) - \gamma'(x + \ell) \rangle = 0,$$

and

$$\|\gamma(x) - \gamma(x + \ell)\|^2 - \ell \langle \gamma(x) - \gamma(x + \ell), -\gamma'(x + \ell) \rangle = 0.$$

If we define $y = (x + \ell)$, then the first of these two equations immediately gives us equation (1.2). Rearranging some terms on the second equation gives us

$$\frac{\|\gamma(x) - \gamma(y)\|}{d(x, y; \gamma)} = -\left\langle \frac{\gamma(x) - \gamma(y)}{\|\gamma(x) - \gamma(y)\|}, \gamma'(y) \right\rangle$$

which, when combined with equation (1.2), gives us equation (1.3) $\square$

We can use Proposition 41 on page 24 to greatly simplify the computation of a curve's distortion. We now provide a couple of examples. While these computations are not rigorous (we use Maple) they illustrate how Proposition 41 can be used to find the distortion of a curve. Consider the ellipse $(a \cos \theta, b \sin \theta)$ and let $\gamma(s)$ denote its parametrization by arclength. It is difficult to find a closed form for the arclength between arbitrary pairs of points on the ellipse. So computing the distortion quotient at all pairs (s, t) with $s \neq t$ is difficult. However, Proposition 41 asserts that the only points x and y that satisfy $dq_\gamma(x, y) = \delta(\gamma)$ must either be antipodal or must also satisfy the equation

$$\langle \gamma(x) - \gamma(y), \gamma'(x) \rangle = \langle \gamma(x) - \gamma(y), \gamma'(y) \rangle .$$

A Maple computation shows that the pairs of points that satisfy Proposition 41 are exactly those that are the intersection of horizontal lines with the ellipse, the intersection of vertical lines with the ellipse, and points that are $\mathrm{Len}(\gamma)/2$ apart along the curve.

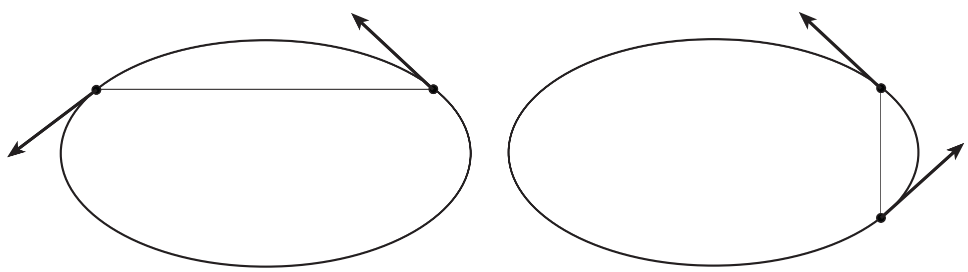

Figure 1.6: Above is an illustration of an example of each of the first two types of critical points. The critical points that are $\mathrm{Len}(\gamma)/2$ apart clearly maximize the distortion quotient at the minor axis.

Assuming that these are the only pairs of points that satisfy Proposition 41, it suffices to consider only them when computing the supremum of the distortion quotient. Another Maple calculation shows that in the case of the ellipse γ, the distortion $\delta(\gamma)$ is achieved at the endpoints of the minor axis. So, if the length of the minor axis is $a/2$, then $\delta(\gamma) = \mathrm{Len}(\gamma)/a$.

Consider now a stadium curve $F\colon [a, b] \longrightarrow \mathbb{R}^2$ illustrated in Figure 1.8 consisting of two half circles of radius R and two line segments of length L connecting the circle arcs. Another Maple calculation shows that there are three types of critical points for dq_F, pairs of points on the straight

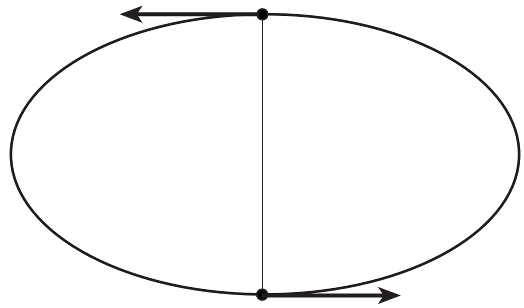

Figure 1.7: The distortion of an ellipse is achieved at the endpoints of the minor axis.

line segments, pairs of points on the circle arcs, and pairs of points that are $\mathrm{Len}(F)/2$ apart along the curve. The first two types are illustrated in Figure 1.8. The distortion quotient for critical points that are $\mathrm{Len}(F)/2$ apart is maximized with the critical points that bisect the straight line segments. Orient the stadium curve so that the center of mass is at the origin and the straight line segments are horizontal. If (x_1, x_2) is any point on one of the straight line segments, then the pair of points (x_1, x_2), $(x_1, -x_2)$ is a critical point for dq_F. In this case, the distortion quotient is maximized when the arclength is maximized since the denominator of the distortion quotient is constantly $2R$. Hence, the largest value for the distortion quotient for these critical points is $(\pi R + L)/2R$ and it occurs when the distortion quotient is evaluated at the pair of points that bisect the straight line segments.

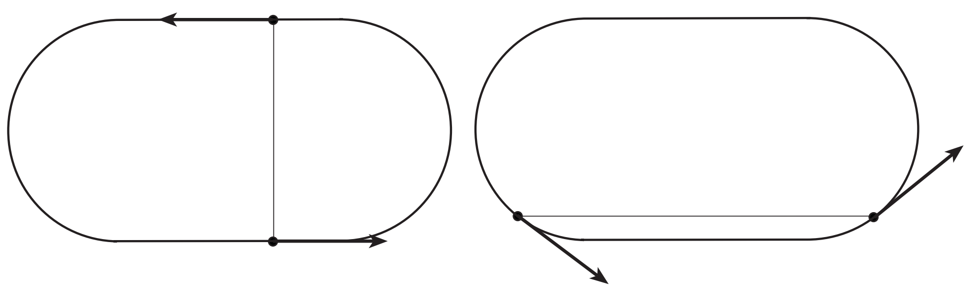

Figure 1.8: Here we see representatives of points that satisfy condition (1) of Proposition 41 for the stadium curve F.

In addition if (y_1, y_2) is any point on one of the circle arcs then the pairs (y_1, y_2), $(y_1, -y_2)$ as well as the pairs $(y_1, y_2), (-y_1, y_2)$ are critical points for dq_F. We know that if the points are on the same circle arc then their distortion quotient can be no more than $\pi/2$. On the other hand, if the pair of points for an arc connecting the two circle arcs, then their distortion quotient varies from 1 to $(\pi R + L)/(2R + L)$. Notice that the latter quantity gets larger as $L \to 0$ and when $L = 0$, its value is $\pi/2$.

Then, assuming the accuracy and completeness of the Maple calculation,

$$\delta(F) = \max\left\{\frac{\pi R + L}{2R}, \frac{\pi R + L}{2R + L}\right\}.$$

These fractions have the same numerator, but the denominator of the second is larger. Thus

$$\frac{\pi R + L}{2R} > \frac{\pi R + L}{2R + L}$$

when $L > 0$ and

$$\delta(F) = \frac{\pi R + L}{2R}.$$

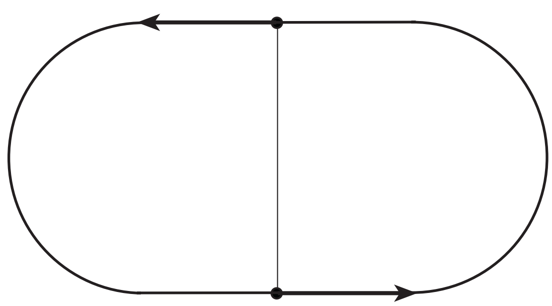

Figure 1.9: The maximum of the distortion quotient for the stadium curve F occurs at the pair of points that bisect the straight line segments.

1.3 Geometry and Distortion

Theorem 42. ***(Crofton)*** *Given any rectifiable plane curve $\gamma\colon [a, b] \longrightarrow \mathbb{R}^2$, let $P_\theta(\gamma)$ be the length of the orthogonal projection of γ onto the line through the origin with direction $(\cos(\theta), \sin(\theta))$.*

Then

$$\operatorname{Len}(\gamma) = \frac{1}{4}\int_0^{2\pi} P_\theta(\gamma)\,\mathrm{d}\theta.$$

That is, the length of γ is a constant multiple of the average length of the projection of γ onto a line through the origin.

Proof. Let Q denote the set of all polygons inscribed in γ. Then by definition, the length of γ is

$$\operatorname{Len}(\gamma) = \sup_{q\in Q} \operatorname{Len} q.$$

Therefore, it will suffice to prove the theorem for each polygon q, since the lengths of the projections of the q converge to the length of the projection of γ. In fact, it suffices to prove the theorem for a single edge of q. Let E be any edge on q with length $\operatorname{Len}(E)$. Without loss of generality we may assume that E has one endpoint at the origin and that E lies on the x-axis.

The projection of E onto the line $t(\cos\theta, \sin\theta)$ has length exactly $\operatorname{Len}(E)|\cos\theta|$. So, averaging this value over all values of $\theta \in [0, 2\pi]$ gives the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Len}(E)|\cos\theta|\,\mathrm{d}\theta = \frac{2\operatorname{Len}(E)}{\pi}.$$

Now, when computing the average projection length of the entire polygon q, it suffices to find the average projection length of each edge individually, and then sum the results. To do so, we view the edges of q as a collection of directed line segments with one vertex at the origin.

Moreover, if an edge E makes an angle ϕ with the positive x-axis, then writing

$$\int_0^{2\pi} P_\theta(E)\,\mathrm{d}\theta = \int_\phi^{2\pi+\phi} P_\theta(E)\,\mathrm{d}\theta$$

we see that we may think of each edge of q as lying on the positive x-axis.

It has now been shown that the length of q is

$$\sum_i \operatorname{Len}(E_i) = \sum_i \frac{1}{4}\int_0^{2\pi} P_\theta(E_i)\,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} P_\theta(\gamma)\,\mathrm{d}\theta.$$

$\square$

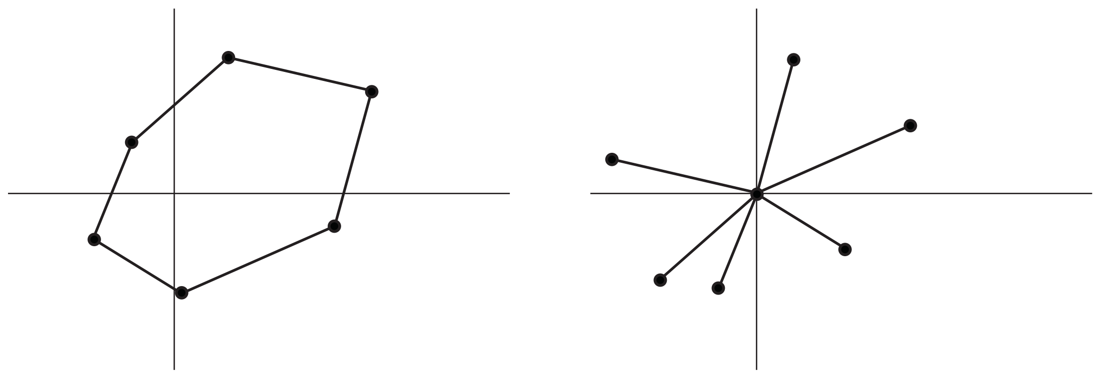

Figure 1.10: On the left we see a polygonal curve. On the right, the edges of the curve after they have been translated so that one endpoint is at the origin. If E is any edge on the curve on the left and $\hat{E}$ is the corresponding edge on the right, then $\mathrm{Len}(E) = \mathrm{Len}(\hat{E})$ and $P_\theta(E) = P_\theta(\hat{E})$ for all θ.

Theorem 43. ***(Schur's Lemma*** *[22]) Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ be a plane arc with total curvature $\kappa([a,b])$ which forms a convex curve with the chord with endpoints $\gamma(a)$ and $\gamma(b)$. Let $\gamma^*\colon [a,b] \longrightarrow \mathbb{R}^3$ be an arc of the same length so that given any interval $[c,d] \subset [a,b]$ we have $\kappa\left([c,d]\right) \geq \kappa^*\left([c,d]\right)$. If ℓ^* and ℓ denote the lengths of the chords joining their endpoints, then $\ell \leq \ell^*$.*

Proof. Let T_γ^+ and $T_{\gamma^*}^+$ be the right tangent indicatrices of γ and γ^* respectively. By convexity there exists $s_0 \in [a,b]$ so that either $T_\gamma^+(s_0)$ is in the same direction as $\gamma(b)-\gamma(a)$ or, T_γ^+ is discontinuous at s_0 and the point $(\gamma(b)-\gamma(a))/\|\gamma(b)-\gamma(a)\|$ lies on the great circle arc with endpoints $T_\gamma^+(s_0)$ and $T_\gamma^-(s_0)$. In either case, let T_0 denote the unit vector in the direction of $\gamma(b)-\gamma(a)$ at the point $\gamma(s_0)$. Again by convexity we know that $\kappa(\gamma([a,b])) \leq 2\pi$. It follows that $\kappa(\gamma([a,s_0])) \leq \pi$ and $\kappa(\gamma([s_0,b])) \leq \pi$.

If the curve γ has no atom for the curvature measure $\mathcal{K}$, i.e., T_γ^+ is continuous at s_0, then γ^* cannot have an atom at s_0. To see this we use the fact that the curvature measure $\mathcal{K}$ is a Radon measure. Let $\mathcal{K}$ and $\mathcal{K}^*$ denote the curvature measures associated to the curves γ and γ^* respectively.

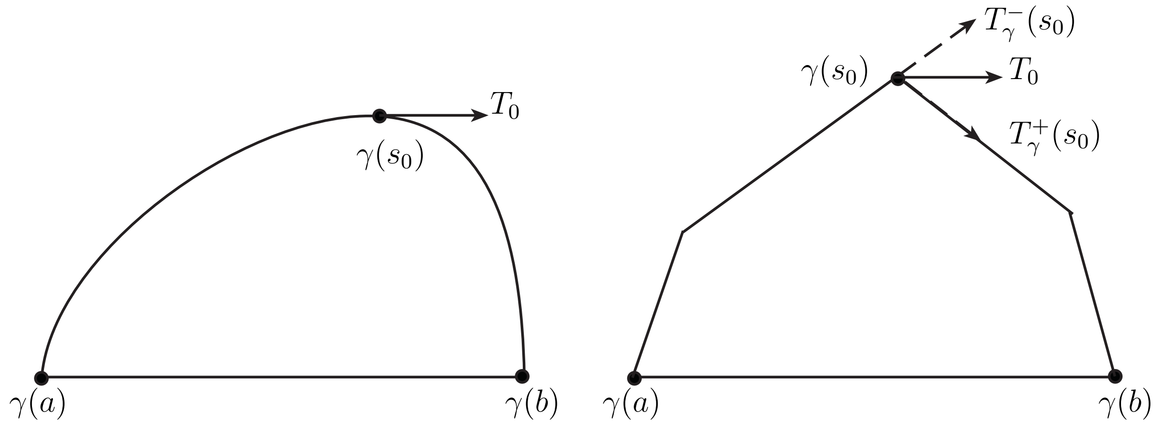

Figure 1.11: Here we see the two cases that can arise when defining T_0. The first case, on the left, occurs when there is a point s_0 so that $T_\gamma^+(s_0)$ is in the direction of the chord joining the endpoints of the arc $\gamma([a,b])$. In the case on the right we see that there is no such s_0. However the convexity of $\gamma([a,b])$ guarantees that there is a point s_0 so that there is a vector in the cone defined by T_γ^- and T_γ^+ that is in the direction of $\gamma(b)-\gamma(a)$.

Since $\mathcal{K}$ is a Radon measure we know

$$\mathcal{K}(\{s_0\}) = \inf\{\mathcal{K}(U) : U \text{ open and } s_0 \in U\}.$$

Now, $\mathcal{K}(\{s_0\}) = 0$ and so there exists a sequence of open sets U_i containing s_0 so that $\mathcal{K}(U_i) \to 0$ as $i \to \infty$. By hypothesis, we also know that $\mathcal{K}^*(U) \leq \mathcal{K}(U)$ for all open intervals U. Therefore $\mathcal{K}^*(U_i) \leq \mathcal{K}(U_i)$ for all i. So,

$$\mathcal{K}^*(\{s_0\}) = \inf\{\mathcal{K}^*(U) : U \text{ open and } s_0 \in U\} \leq \lim_{i\to\infty} \mathcal{K}^*(U_i) \leq \lim_{i\to\infty} \mathcal{K}(U_i) = 0.$$

In this case we rigidly rotate γ^* so that $T_{\gamma^*}^+(s_0) = T_0$.

If instead T_γ^+ is discontinuous at s_0 then, using a similar measure theoretic argument, $T_{\gamma^*}^+$ is also discontinuous at s_0. In this case we rotate γ^* so that

$$d(T_\gamma^+(s_0), T_0; S^2) \geq d(T_{\gamma^*}^+(s_0), T_0; S^2)$$

and

$$d(T_\gamma^-(s_0), T_0; S^2) \geq d(T_{\gamma^*}^-(s_0), T_0; S^2).$$

We have chosen T_0 in this way so that we can express the length of the chords as

$$\|\gamma(b) - \gamma(a)\| = \langle \gamma(b) - \gamma(a), T_0 \rangle = \int_a^b \left\langle T_\gamma^+(s), T_0 \right\rangle ds,$$

and

$$\|\gamma^*(b) - \gamma^*(a)\| \geq \langle \gamma^*(b) - \gamma^*(a), T_0 \rangle = \int_a^b \left\langle T_{\gamma^*}^+(s), T_0 \right\rangle ds.$$

Now it remains only to show that $\left\langle T_\gamma^+(s), T_0 \right\rangle \leq \left\langle T_{\gamma^*}^+(s), T_0 \right\rangle$ almost everywhere.

Observe that

$$\left\langle T_0, T_\gamma^+(s) \right\rangle = \cos\left(d(T_0, T_\gamma^+(s); S^2) \right),$$

and

$$\left\langle T_0, T_{\gamma^*}^+(s) \right\rangle = \cos\left(d(T_0, T_{\gamma^*}^+(s); S^2) \right)$$

since all the points T_0, $T_\gamma^+(s)$, and $T_{\gamma^*}^+(s)$ lie on a unit sphere. Moreover, $\cos(x)$ is an non-increasing function on the interval $[0, \pi]$ and so it will suffice to prove

$$d(T_0, T_{\gamma^*}^+(s); S^2) \leq d(T_0, T_\gamma^+(s); S^2).$$

In addition since $\gamma([s, s_0])$ is a planar arc of total curvature at most π we know T_γ^+ is a single great circle arc of length at most π. Thus

$$d(T_0, T_\gamma^+(s); S^2) = \mathrm{Len}_{S^2}(T_\gamma^+([s, s_0])).$$

We know that $d(T_0, T_{\gamma^*}^+(s); S^2) \leq \mathrm{Len}_{S^2}(T_{\gamma^*}([s, s_0]))$. Therefore it remains only to show that $\mathrm{Len}_{S^2}\left(T_{\gamma^*}^+([s, s_0])\right) \leq \mathrm{Len}_{S^2}(T_\gamma^+([s, s_0]))$. But, by definition of curvature measure for finite total curvature curves as defined by the right continuous function $F_\mathcal{K}$ in Definition 34 on page 18, these lengths are $\kappa([s, s_0])^*$ and $\kappa([s, s_0])$ respectively and by hypothesis $\kappa([s, s_0])^* \leq \kappa([s, s_0])$. This completes the proof. $\square$

In their 1998 paper [1], Robert Kusner and John Sullivan considered rectifiable curves and studied the relationship between a closed curve's distortion and its ropelength. The ropelength of a curve is the ratio of the length of the curve to the curve's "thickness". The thickness of a curve measures the minimum of the lengths of a certain family of chords on the curve. We cannot include all chords in the family, since the lengths of chords joining nearby points on the curve must be small. We can avoid this problem when computing thickness by ignoring pairs whose distortion is less than some threshold. Following Kusner and Sullivan in [1], we give

Definition 44. *Given any real number $b \geq 1$, the* b ***distortion thickness*** *of a curve $\gamma \subset \mathbb{R}^n$ is*

$$\tau_b(\gamma) \colon = \inf_{dq_\gamma(p,q) \geq b} \|\gamma(p) - \gamma(q)\|,$$

the minimum distance between points of distortion more than b.

The following proposition from [1] is of direct interest here. We define antipodal points on a closed curve γ to be points that are $\mathrm{Len}(\gamma)/2$ apart along the curve.

Proposition 45. *Let $\gamma \subset \mathbb{R}^n$ be any closed rectifiable curve. Define $\delta_0(\gamma)$ to be the supremum of the distortion quotient taken over all pairs of* antipodal *points. Then, $\delta(\gamma) \geq \delta_0(\gamma) \geq \pi/2$, with equality if and only if γ is a round circle.*

Proof. Since the distortion is scale invariant, we can assume that γ has length 2π. Further assume that γ is parametrized by arclength. It is immediate that $\delta(\gamma) \geq \delta_0(\gamma)$ since each is computed as a supremum of the distortion quotient and the supremum used to compute $\delta(\gamma)$ is taken over a larger set. Therefore, it suffices to find a pair of antipodal points $\gamma(s)$ and $\gamma(s+\pi)$ so that $\|\gamma(s) - \gamma(s+\pi)\| \leq 2$. Indeed, if such a pair of points exists then, since we know $\gamma(s)$ and $\gamma(s+\pi)$ are antipodal, we have,

$$\delta_0(\gamma) \geq \frac{\pi}{\|\gamma(s) - \gamma(s+\pi)\|} \geq \frac{\pi}{2}.$$

Assume that no such pair $\gamma(s)$ and $\gamma(s+\pi)$ exists. Then all pairs of antipodal points must be a distance more than 2 apart. Define a new curve f by $f(s) = \gamma(s) - \gamma(s+\pi)$. Our assumption is now that f lies completely outside $B_2(0)$, the closed ball of radius 2 in $\mathbb{R}^n$. Notice that this function has the following properties:

(1) $f(s+\pi) = \gamma(s+\pi) - \gamma((s+\pi)+\pi) = -(\gamma(s) - \gamma(s+\pi)) = -f(s)$, and

(2) since γ is parametrized by arclength we have

$$\begin{aligned}\|f(s) - f(t)\| &= \|\gamma(s) - \gamma(s+\pi) - \gamma(t) + \gamma(t+\pi)\| \\ &\leq \|\gamma(s) - \gamma(t)\| + \|\gamma(t+\pi) - \gamma(s+\pi)\| \\ &\leq |s-t| + |t+\pi-s-\pi| \\ &= 2|t-s|.\end{aligned}$$

That is, f is Lipschitz with speed at most 2.

The first of these conditions tells us that any arc A of f from $f(s)$ to $f(s+\pi)$ has endpoints that are antipodal points on a sphere of radius $\|f(s)\|$ centered at the origin in $\mathbb{R}^n$.

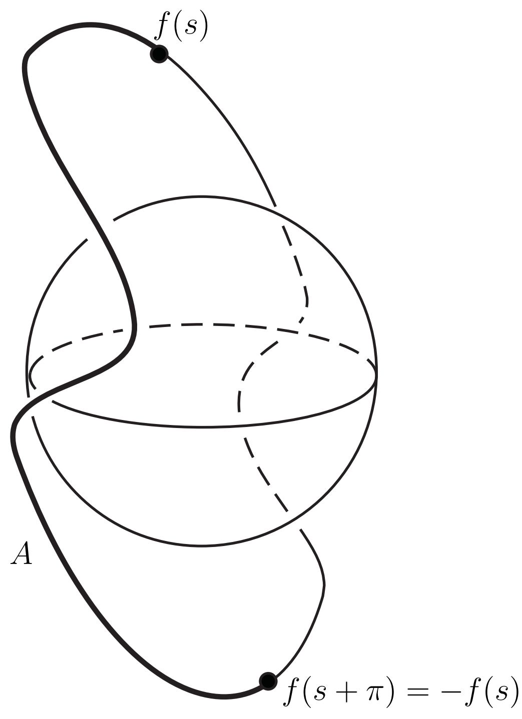

Figure 1.12: Illustration of condition (1) in the proof of Proposition 45 on the preceding page.

So, now let us consider the length of the arc A. Since the arc lies completely outside the closed ball of radius 2, its length must be more than the length of a geodesic connecting two antipodal points on $\partial B_2(0)$, which is 2π. But, condition (2) above stipulates that the arc from $f(s)$ to $-f(s) = f(s+\pi)$ can have length no more than $2((s+\pi)-s) = 2\pi$, a contradiction.

Note that we can achieve a distortion of $\pi/2$ exactly when the curve f traces out a great circle. This completes the proof. ☐

Kusner and Sullivan use this lemma to show that the $\tau_{\pi/2}$ thickness is well defined for any closed curve. This thickness will not quite serve our purposes, and instead we will rely later on the $\tau_{\delta([\gamma])}$ thickness for curves of knot type $[\gamma]$.

We will also need a lemma from Denne and Sullivan in [10].

Lemma 46. *An arc $\gamma_{ab} \in \mathbb{R}^n$ of total curvature $\kappa < \pi$ has distortion $\delta(\gamma_{ab}) \leq \sec \kappa/2$.*

Proof. Let $\gamma(a)$ and $\gamma(b)$ denote the endpoints of γ_{ab}. Consider the planar curve $\hat{\gamma}_{ab}$ with the same length and the same curvature measure as γ_{ab} which forms a convex curve with its chord, and let $\hat{\gamma}(a)$ and $\hat{\gamma}(b)$ denote its endpoints. Schur's lemma asserts that $d(\hat{\gamma}(a), \hat{\gamma}(b); \mathbb{R}^3) \leq d(\gamma(a), \gamma(b); \mathbb{R}^3)$. Then, for all $s, t \in [a, b]$

$$dq_{\hat{\gamma}}(s,t) = \frac{d(\hat{\gamma}(s), \hat{\gamma}(t); \hat{\gamma})}{d(\hat{\gamma}(s), \hat{\gamma}(t); \mathbb{R}^3)} = \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\hat{\gamma}(s), \hat{\gamma}(t); \mathbb{R}^3)} \geq \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)} = dq_{\gamma}(s,t).$$

Pick any two points x and z in the plane. Given an initial direction for an edge starting at x there is a unique position for a point y on that edge forming a triangle xyz with exterior angle κ at y. If we let $x = \hat{\gamma}(a)$ and $z = \hat{\gamma}(b)$, then the entire arc $\hat{\gamma}_{ab}$ must lie within the triangle xyz. Crofton's formula implies that $\hat{\gamma}_{ab}$ must be shorter than the polygonal segment $xy \cup yz$. So, $dq_{\hat{\gamma}}(a,b) \leq \delta(xy \cup yz)$.

We must now compute $\delta(xy \cup yz)$. We may assume that $y = (0,0)$, $x = (-p, q)$, and $z = (p, r)$ as in Figure 1.13. The segment $xy \cup yz$ is differentiable everywhere except at the corner point. Hence Proposition 41 tells us that the only candidates for points away from y that realize the distortion occur when the two angles θ and ϕ in Figure 1.13 are equal. In this case, the resulting triangle with angles θ, ϕ and $\pi - \kappa$ is isoceles and so the distortion quotient is $\sec(\kappa/2)$. For pairs including y the distortion quotient is 1. So, since $\sec(\kappa/2) \geq 1$ we have $\delta(xy \cup yz) = \sec(\kappa/2)$.

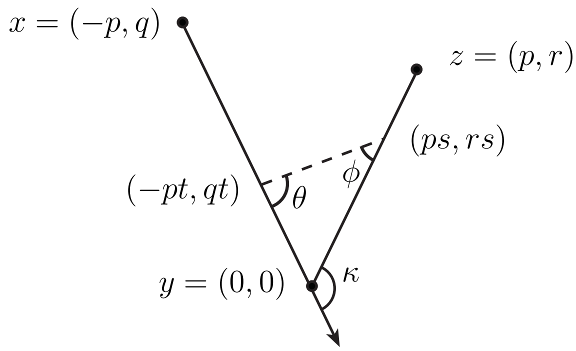

Figure 1.13: It suffices to consider a triangle with vertices $(-p, q)$, $(0, 0)$, and (p, r).

Now, suppose $\hat{\gamma}(s)$ and $\hat{\gamma}(t)$ are any points on the arc $\hat{\gamma}_{ab}$. If we let κ' denote the total curvature on the arc $\hat{\gamma}_{st}$, then $\kappa' \leq \kappa$. By repeating the argument above we know that $dq_{\hat{\gamma}}(s, t) \leq \sec(\kappa'/2)$ and since $\sec(\phi)$ is an increasing function on the interval $(0, \pi)$ it follows that $dq_{\hat{\gamma}}(s, t) < \sec(\kappa/2)$. Thus

$$\delta(\gamma_{ab}) \leq \delta(\hat{\gamma}_{ab}) \leq \delta(xy \cup yz) = \sec\left(\frac{\kappa}{2}\right)$$

and this completes the proof.

$\square$

Chapter 2

Main Theorem

2.1 A New Point Of View On Gromov's Distortion

In [4] Gromov defines the distortion of a continuous curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ as

$$\delta(\gamma) := \sup_{s\neq t} \frac{d(\gamma(s),\gamma(t);\gamma)}{d(\gamma(s),\gamma(t);\mathbb{R}^3)}.$$

where the fraction is called the distortion quotient $dq_\gamma(s,t)$. He then poses the question, "Does there exist a curve γ in each knot type with $\delta(\gamma) < 100$?" The question really asks whether or not there exists a uniform upper bound for the distortion of all knot types. Though it has attracted considerable interest, Gromov's question has proved very difficult to answer.

It is natural to conjecture that no such bound exists. A proof would require two steps. We first need to find a topological invariant $\mathcal{X}$ that increases with knot complexity. We must then construct a lower bound on the knot's distortion in terms of $\mathcal{X}$. Using this lower bound, we can then exhibit a family of knots $\{\gamma_i\}$ so that the sequence $\{\delta([\gamma_i])\}$ diverges. Neither step seems easy to carry out.

Choosing a suitable invariant $\mathcal{X}$ requires care. Our topological invariant must increase without bound for some, but not all, families of knots. For a smooth knot γ there exists an upper bound for the sequence $\{\delta([\gamma]),\delta([\gamma\#\gamma]),\delta([\gamma\#\gamma\#\gamma]),\ldots\}$[1]. It follows that if the sequence $\{\mathcal{X}([\gamma]),\mathcal{X}([\gamma\#\gamma]),\mathcal{X}([\gamma\#\gamma\#\gamma]),\ldots\}$ diverges, then $\mathcal{X}$ will be of no use when defining a divergent sequence of lower bounds for distortion. This rules out many well known invariants such as crossing number, bridge number, and genus.

[1] This was first pointed out in print by O'Hara [7], who observed that the distortion of a number of tiny knots arranged around a large circle is independent of the number of knots.

The next task involves constructing a lower bound on the distortion of a curve in terms of the chosen topological invariant. This requires finding points $\gamma(s)$ and $\gamma(t)$ on the curve γ for which $dq_\gamma(s,t)$ is large. Unfortunately, such points can be quite elusive.

The main theorem shows that length minimizing curves in a particular subset $U_C([\gamma])$ of $[\gamma]$ have the property that points with relatively large distortion quotient saturate arcs of positive total curvature. This should make it easier to bound the distortion of curves in $U_C([\gamma])$ in terms of a topological invariant $\mathcal{X}([\gamma])$.

The set $U_C([\gamma])$ has several membership requirements in addition to the fact that $U_C([\gamma]) \subset [\gamma]$. For technical reasons we assume all curves in $U_C([\gamma])$ have finite total curvature. Second, we require that a constant C bounds the distortion of knots in $U_C([\gamma])$ from above. If we choose C to be a constant multiple of $\delta([\gamma])$, this bound will allow us to relate lower bounds for distortion of curves in $U_C([\gamma])$ to lower bounds on $\delta([\gamma])$. Hence any divergent sequence of lower bounds $\{b_i\}$ on the distortion of knots in, say, $\{U_{2\delta([\gamma])}([\gamma_i])\}$ will provide a negative answer to Gromov's question. This is illustrated in Figure 2.1 on the next page.

The last requirement for membership in $U_C([\gamma])$ stems from our interest in studying curves of minimum length. Due to the scale invariance of distortion the set of length minimizers in $U_C([\gamma])$ will be empty unless we add another constraint. Hence we must fix a scale for curves in $U_C([\gamma])$. But even if we prevented a sequence of curves $\{\gamma_k\} \subset U_C([\gamma])$ approaching a curve γ_0 of infimal length from shrinking to a point, we would still be concerned that γ_0 might not be in $[\gamma]$. After all, knotted regions can pull tight when decreasing length as seen in Figure 2.2 on page 40. To prevent this, we fix scale in a carefully chosen way. Recall the b-distortion thickness of a curve γ is given by

$$\tau_b(\gamma) = \inf_{dq_\gamma(p,q)\geq b} \|\gamma(p) - \gamma(q)\|,$$

the infimal distance between pairs of points on γ with distortion quotient at least b. We require that $\tau_{\delta([\gamma])}(\gamma) \geq 1$ for curves $\gamma \in U_C([\gamma])$.

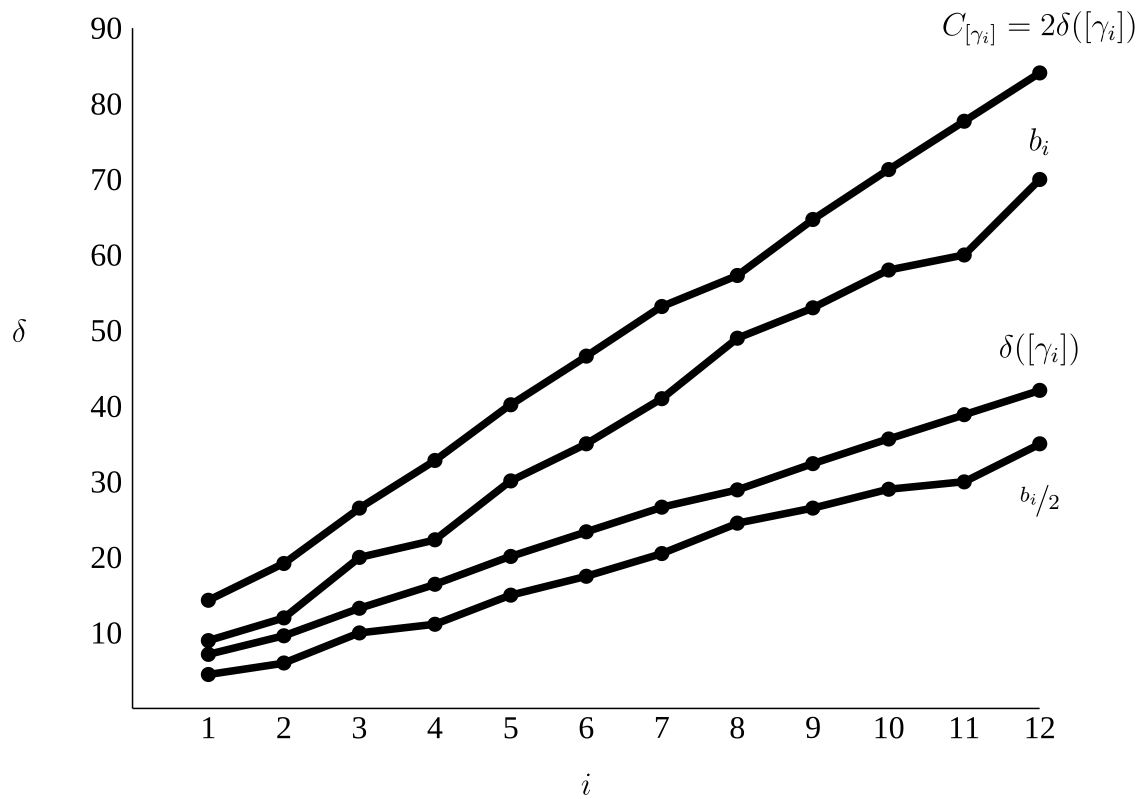

Figure 2.1: Here we see a graph representing a divergent sequence $\{b_i\}$ of lower bounds on the distortion of certain curves in $U_{C_{\gamma_i}}([\gamma_i])$. If we define $C_{\gamma_i} = 2\delta([\gamma_i])$ then this will also yield a divergent sequence $\{b_i/2\}$ of lower bounds on $\delta([\gamma_i])$.

We are now prepared to state the main thereom. For now we think of a "$\delta([\gamma])$-drc" as a pair of points (s, t) for which $dq_\gamma(s, t) = \delta([\gamma])$. We will define this term precisely in Definition 50 on page 44.

Theorem 55 (Main Theorem). *Let $U_C([\gamma])$ be the set of all finite total curvature curves γ in $[\gamma]$, with distortion $\delta(\gamma) < C$ and distortion thickness $\tau_{\delta([\gamma])} \geq 1$ (i.e., any point (s, t) with $dq_\gamma(s, t) \geq \delta([\gamma])$ satisfies $\|\gamma(s) - \gamma(t)\| \geq 1$). Then any open interval on a curve of minimum length in $U_C([\gamma])$ is either a straight line segment or contains an endpoint of a $\delta([\gamma])$-drc.*

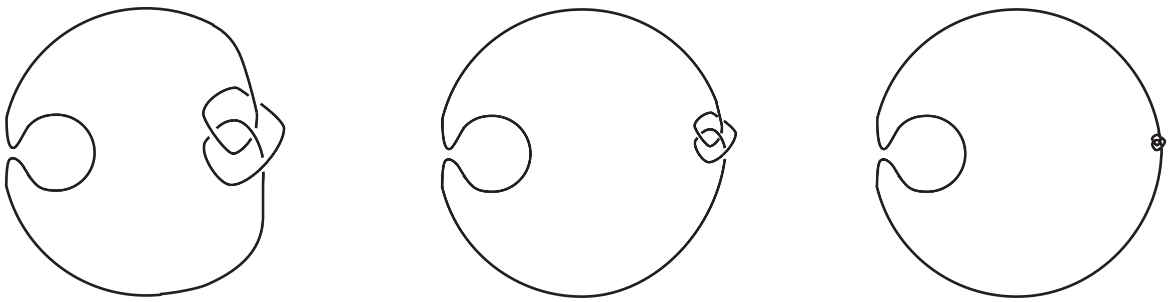

Figure 2.2: Here are three representatives from a sequence of figure eight knots with decreasing length that converges to the unknot. The pinch on the left illustrates the concern that every member of the sequence of curves may have maximum distortion realized by a pair of points a constant distance apart even though the knotted region is shrinking to a point.

The proof of the main theorem will be a proof by contradiction. Suppose that we have a curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ of minimum length in $U_C([\gamma])$ and an arc $\gamma((c,d))$ on γ with positive total curvature that does not contain an endpoint of a $\delta([\gamma])$-drc. We show that it is possible to decrease the length of the arc $\gamma((c,d))$ to obtain a new curve that is also a member of $U_C([\gamma])$. We have then reached a contradiction since γ is a curve of minimum length in $U_C([\gamma])$. The proof requires two propositions.

(1) There exists $\varepsilon > 0$ so that $dq(s,t) < \delta([\gamma]) - \varepsilon$ for all points $(s,t) \in (c,d) \times [a,b] \cup [a,b] \times (c,d)$. (Proposition 52 on page 45)

(2) The arc $\gamma((c,d))$ can be changed in a length decreasing way so that the increase of the distortion quotient on $(c,d) \times [a,b] \cup [a,b] \times (c,d)$ is less than ε and the change of the distortion quotient on $(c,d) \times (c,d) - ((c,d) \times [a,b] \cup [a,b] \times (c,d))$ is nonpositive. Hence the distortion of γ is not increased. (Proposition 53 on page 46)

We point out that if dq_γ could be extended to a continuous function on $[a,b] \times [a,b]$ then (1) would be immediate and (2) would be much easier to prove. However, this is not always possible for finite total curvature curves.

2.2 On the Discontinuity of dq_γ for $\gamma \in \mathrm{FTC}$

Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a continuous embedded curve. Then since γ is continuous, the functions $d(\cdot,\cdot;\gamma)$ and $d(\cdot,\cdot;\mathbb{R}^3)$, restricted to points on γ, are each continuous functions. So their ratio, the distortion quotient, is continuous whenever the denominator is nonzero. Indeed, dq_γ is only defined for points off the diagonal of $[a,b]\times[a,b]$.

Regrettably, if $\gamma \in \mathrm{FTC}$ then it may not be possible to define dq_γ on the diagonal in a way that will result in a continuous function. While curves in FTC have one-sided tangents everywhere, the right and left tangent may not be equal at a given point s_0. This can cause a non-removable discontinuity of dq_γ at (s_0, s_0). The following example illustrates such a scenario.

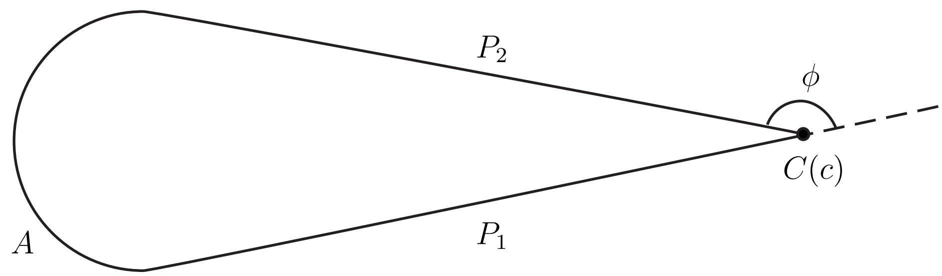

Figure 2.3: This example illustrates a discontinuity of the distortion quotient. The discontinuity of the tangent curve at the point corresponding to the corner point causes a jump in the distortion quotient.

Let $\mathcal{C}\colon [a,b] \longrightarrow \mathbb{R}^2$ denote the comet shaped curve in Figure 2.3 consisting of an arc A of a circle and a polygonal section, consisting of two line segments P_1 and P_2, with exterior angle ϕ. Let $c \in [a,b]$ so that $\mathcal{C}(c)$ is the corner point. A calculation in the proof of Lemma 46 on page 35 shows that $\delta(\mathcal{C}) = \sec(\phi/2)$ and $dq_{\mathcal{C}}(s,t) = \delta(\mathcal{C})$ on the continuum of points

$$E = \{(s,t) \in [a,b]\times[a,b] : \mathcal{C}(s) \in P_1,\ \mathcal{C}(t) \in P_2, \text{ and } d(\mathcal{C}(s),\mathcal{C}(c),\mathcal{C}) = d(\mathcal{C}(t),\mathcal{C}(c),\mathcal{C})\}.$$

Now, if $\{(s_i,t_i)\} \subset E$ is a sequence so that $(s_i,t_i) \to (c,c)$ as $i \to \infty$, then

$$\lim_{i\to\infty} dq_{\mathcal{C}}(s_i,t_i) = \sec\frac{\phi}{2}.$$

On the other hand, for all points s so that $\mathcal{C}(s)$ is a point on either line segment, we have $dq_{\mathcal{C}}(s,c) = 1$, hence

$$\lim_{s\to c} dq_{\mathcal{C}}(s,c) = 1.$$

We can readily see that the discontinuity along the diagonal of $[a,b]\times[a,b]$ is not removable. In fact, since the distortion is defined as a supremum, it may be the case that for a general curve $\gamma \in \mathrm{FTC}$, there exists a sequence of points (s_i,t_i) converging to a point (s,s) so that $dq_\gamma(s_i,t_i) \to \delta(\gamma)$, $dq_\gamma(s,t) < \delta(\gamma)$ everywhere dq_γ is defined, and dq_γ has a non-removable discontinuity at (s,s). Curves of this type have no distortion realizing chord. An example of such a curve is the *Dragon's tooth curve* in Figure 2.4.

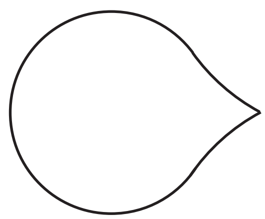

Figure 2.4: The distortion of this curve is realized as a limit of distortion quotients of pairs of symmetric points converging to the corner point. The parameter space $[a,b]\times[a,b]$ has no point (s,t) that realizes the distortion.

This curve, $\mathcal{D}\colon [a,b] \longrightarrow \mathbb{R}^2$, is formed by three circle arcs: one of small radius r and two of larger radius R. The large circle arcs are connected to the smaller circle arc so that their tangents agree at the points of intersection. The large circles meet at a corner with exterior angle $\phi < \pi$. The distortion of a circle is $\pi/2$, so if $\mathcal{D}(s)$ and $\mathcal{D}(t)$ are points on the same circle arc then $dq_{\mathcal{D}}(s,t) \le \pi/2$. If $\mathcal{D}(s)$ lies on one of the large circle arcs and if $\mathcal{D}(t)$ lies on the small circle arc, then $dq_{\mathcal{D}}(s,t) \to 1$ as $d(\mathcal{D}(s),\mathcal{D}(t);\mathbb{R}^2) \to 0$. Therefore, the distortion quotient cannot grow large because the denominator becomes small. On the other hand, the numerator is no larger than the length of the two arcs. So $dq_{\mathcal{D}}(s,t)$ is uniformly bounded above for all points (s,t) for which $\mathcal{D}(s)$ and $\mathcal{D}(t)$ lie on circle arcs of different radius.

Let $\mathcal{D}(c)$ denotes the corner point. Let $\{(s_i, t_i)\}$ be a sequence so that $\mathcal{D}(s_i)$ is on one large circle arc, $\mathcal{D}(t_i)$ is on the other large circle arc and $d(\mathcal{D}(s_i), \mathcal{D}(c); \mathcal{D}) = d(\mathcal{D}(t_i), \mathcal{D}(c); \mathcal{D})$ for all i. If $(s_i, t_i) \to (c, c)$ then

$$\lim_{i\to\infty} dq_{\mathcal{D}}(s_i, t_i) = \sec\frac{\phi}{2}.$$

Forcing ϕ to be close to π in the construction of $\mathcal{D}$ forces $dq_{\mathcal{D}}(s,t) < \sec\phi/2$ for all (s,t) in the domain of $dq_{\mathcal{D}}$. So, the distortion of $\mathcal{D}$ is evaluated as a limit of $dq_{\mathcal{D}}(s,t)$ as (s,t) converges to a point on the diagonal that is a non-removable discontinuity.

To deal with examples like this, the definition of a distortion realizing chord will need to be extended. Just considering pairs of points (s,t) so that $dq_\gamma(s,t) = \delta(\gamma)$ is not sufficient. After all, there may be no such pairs. We start with a definition.

Definition 47. *For any* $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ *in* FTC*, we define the function* $D_\gamma\colon [a,b] \longrightarrow \mathbb{R}$ *by*

$$D_\gamma(s)\colon = \sup_t dq_\gamma(s,t)$$

Lemma 48. *If* $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ *is a curve with finite total curvature parametrized by arclength, then*

$$D_\gamma(s) = \sup_t \frac{d(\gamma(s),\gamma(t);\gamma)}{d(\gamma(s),\gamma(t);\mathbb{R}^3)} = \max_t \frac{d(\gamma(s),\gamma(t);\gamma)}{d(\gamma(s),\gamma(t);\mathbb{R}^3)}.$$

Proof. One-sided derivatives exist everywhere by Proposition 27 on page 12, hence

$$\lim_{t\to s^+} \frac{\gamma(t)-\gamma(s)}{t-s}$$

exists. We know, by the continuity of the norm function $\|\cdot\|$, that

$$\lim_{t\to s^+} \left\| \frac{\gamma(t)-\gamma(s)}{t-s} \right\| = \lim_{t\to s^+} \frac{\|\gamma(t)-\gamma(s)\|}{|t-s|}$$

exists and equals 1 since the curve is arclength parametrized. So,

$$\lim_{t\to s^+} \frac{|t-s|}{\|\gamma(t)-\gamma(s)\|} = 1.$$

But this is exactly the one-sided limit of the distortion quotient function $dq_\gamma(s,t)$ if we leave s fixed. A similar computation shows that the limit from the left exists as well and is also equal to 1. Therefore, since the function $dq_\gamma(s,t)$ with s fixed is a continuous function on the compact set $[a,b]$, it follows that it achieves its maximum at a point in $[a,b]$. $\square$

In the case of the Dragon's tooth $\mathcal{D}$, the function $D_{\mathcal{D}}(s)$ is defined at the corner point, but it is less than the distortion of the curve. However, we do have $D_{\mathcal{D}}(s_i) \to \delta(\mathcal{D})$ as $s_i \to s$. We will redefine D_γ at such points so that our new function will achieve its supremal value on γ.

Definition 49. *If $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ is a curve with finite total curvature parametrized by arclength, then we define the **distortion shadow**, denoted $\overline{D}_\gamma(s)$, as the upper envelope of $D_\gamma(s)$. That is*

$$\overline{D}_\gamma(s)\colon = \inf_{\varepsilon>0} \sup_{|s-x|<\varepsilon} \left(\max_t \frac{d(\gamma(x),\gamma(t);\gamma)}{d(\gamma(x),\gamma(t);\mathbb{R}^3)} \right).$$

We are now in a position to make the definition of a distortion realizing chord precise. The Dragon's tooth example shows that we will need to define the distortion quotient along the diagonal in a way that guarantees that the value of $dq_\gamma(s,s)$ is defined to be the largest limiting value of all sequences approaching it.

Definition 50. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a finite total curvature curve parametrized by arclength. If (s,t) is such that $s \neq t$ and*

$$\overline{dq_\gamma}(s,t) := \inf_{\varepsilon>0} \sup_{\|(x,y)-(s,t)\|<\varepsilon} \frac{d(\gamma(x),\gamma(y);\gamma)}{d(\gamma(x),\gamma(y);\mathbb{R}^3)} \geq k,$$

*then we say (s,t) (or the chord with endpoints $\gamma(s)$ and $\gamma(t)$) is a k-**distortion realizing chord** (k-**drc**).*

Lemma 51. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a finite total curvature curve parametrized by arclength. Then $\overline{D}_\gamma(s) \geq k$ if and only if there exists a value t so that the chord with endpoints $\gamma(s)$ and $\gamma(t)$ is a k-drc.*

Proof. First assume that there is a value t so that the point (s,t) defines a k-drc. We know that there exists a sequence $\{(s_i,t_i)\}$ so that $(s_i,t_i) \to (s,t)$ and $dq_\gamma(s_i,t_i) \to K \geq k$. Since the function $D_\gamma(s_i)$ computes the maximum over all values of t, it is evident that $D_\gamma(s_i) \geq dq_\gamma(s_i,t_i)$. Furthermore, Lemma 5 on page 3 shows us that $\overline{D}_\gamma(s_i) \geq D_\gamma(s_i)$. So, our string of inequalities then becomes

$$\overline{D}_\gamma(s) \geq \lim \overline{D}_\gamma(s_i) \geq \lim D_\gamma(s_i) \geq \lim dq_\gamma(s_i,t_i) = K \geq k.$$

The reverse implication is immediate from the definitions. Let $\{s_i\} \subset [a,b]$ be a sequence so that $D_\gamma(s_i) \to K \geq k$. For each s_i there exists a t_i so that $D(s_i) = dq_\gamma(s_i,t_i)$ by Lemma 48. Furthermore, since the points (s_i,t_i) are elements of the compact set $[a,b] \times [a,b]$ we may assume, by restricting to a subsequence, that $(s_i,t_i) \to (s,t)$ for some $(s,t) \in [a,b] \times [a,b]$. It remains to show $\overline{dq}_\gamma(s,t) \geq k$. Indeed, for any $\varepsilon > 0$

$$\sup_{\|(x,y)-(s,t)\|<\varepsilon} dq(x,y) \geq k$$

since the set on which the supremum is taken contains infinitely many elements of the sequence $\{(s_i,t_i)\}$. Therefore,

$$\overline{dq}_\gamma(s,t) = \inf_{\varepsilon>0} \sup_{\|(x,y)-(s,t)\|<\varepsilon} dq(s,t) \geq k.$$

So the pair (s,t) defines a k-drc. $\square$

By defining k-drc's in terms of an upper semicontinuous function we have made the proof of the first step towards the main theorem relatively simple. Indeed, upper semicontinuous functions achieve their maximum value on compact sets. Therefore, if a closed arc is void of endpoints of k-drcs, then the maximum value of $\overline{dq_\gamma}$ is bounded away from k by a positive quantity ε. Hence, the distortion quotient is bounded away from k by a positive quantity. We make this rigorous below.

Proposition 52. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve of finite total curvature. If $(c,d) \subset [a,b]$ so that $\overline{dq_\gamma}(s,t) < k$ for all $(s,t) \in (c,d) \times [a,b] \cup [a,b] \times (c,d)$ then, for any subinterval $[p,q] \subset (c,d)$ there exists an $\varepsilon > 0$ so that $dq_\gamma(s,t) \leq k - \varepsilon$ for all $(s,t) \in (p,q) \times [a,b] \cup [a,b] \times (p,q)$*

Proof. By assumption the interval (c,d) is free from endpoints of k-drc's. Then using Lemma 51 on page 44, it follows that for every $s \in (c,d)$ the value of $\overline{D}_\gamma(s) < k$. Let $[p,q]$ be any closed interval subset of (c,d). Since $\overline{D}_\gamma(s)$ is upper semicontinuous and bounded, $\overline{D}_\gamma(s)$ has a maximum M on $[p,q]$ by Lemma 5 on page 3. We can let $\varepsilon = k - M$. ☐

2.3 Decreasing Length Without Increasing Distortion

Now that we know that arcs of γ void of k-drc's have distortion bounded away from $\delta(\gamma)$ by some positive quantity ε, it remains to show the length of an arc can be decreased while changing the distortion quotient by an amount smaller than ε. This will complete the second step in the proof of the main theorem outlined in Section 2.1.

Proposition 53. *Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a finite total curvature curve and suppose $\gamma((c,d))$ is any arc with nonzero total curvature. Then, given $\varepsilon > 0$, we can replace $\gamma((c,d))$ with an arc $P((c,d))$ of shorter length so that $dq_P(s,t) - dq_\gamma(s,t) < \varepsilon$ for all $(s,t) \in (c,d) \times [a,b] \cup [a,b] \times (c,d)$.*

Proof. Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve parametrized by arclength and suppose that $\gamma((c,d))$ is an arc of γ of length $L = d - c$ with nonzero total curvature. Let $\varepsilon > 0$ be given. This proof has several cases. We will first assume that we can find a subarc $\gamma((c,d))$ whose total curvature is small. Without loss of generality we assume this is the arc $\gamma((c,d))$. We will then modify only a subarc $\gamma([\hat{c},\hat{d}])$ of $\gamma((c,d))$. To prove that distortion has not greatly increased, we will then examine the change in the distortion quotient. Pairs of points s and t which are relatively close will require a different argument than pairs where s and t are relatively far apart. Figure 2.5 may help the reader to understand the different components of the proof.

Case 1. *Assume the arc $\gamma((c,d))$ has total curvature $0 < K < 2\cos^{-1}(1/(1+\varepsilon/2))$; then by Lemma 46 on page 35, the distortion of the arc $\gamma((c,d))$ is no more than $1+\varepsilon/2$.*

Proof. By replacing the interval $\gamma([c,d])$ with an inscribed polygon, we can decrease length and not increase total curvature. By Lemma 46 on page 35, since the total curvature has not increased,

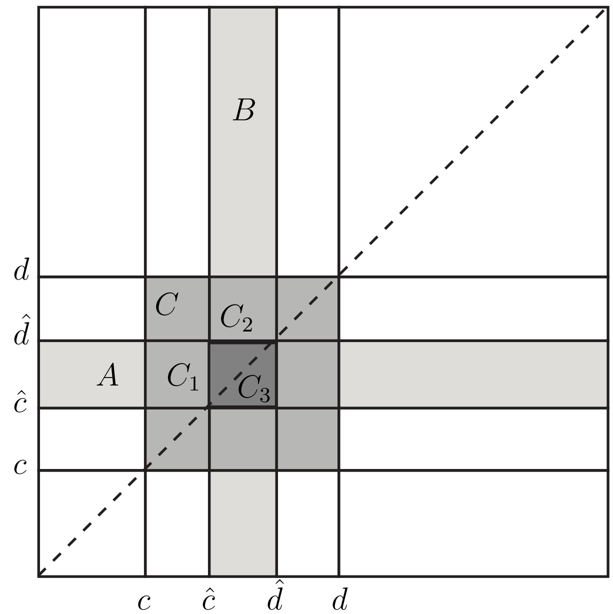

Figure 2.5: We break up our proof into several pieces according to a subdivision of the parameter space. Define $A := [a, c] \times [\hat{c}, \hat{d}]$, $B := [\hat{c}, \hat{d}] \times [d, b]$, $C := [c, d] \times [c, d]$, $C_1 := [c, \hat{c}] \times [\hat{c}, \hat{d}]$, $C_2 := [\hat{c}, \hat{d}] \times [\hat{d}, d]$, and $C_3 := [\hat{c}, \hat{d}] \times [\hat{c}, \hat{d}]$. We use Lemma 46 on page 35 to control the change in the distortion quotient inside the region C. Then, we restrict our attention to the regions A and B, which are bounded away from the diagonal of $[a, b] \times [a, b]$, and use a more straightforward calculation to show that the change in the distortion quotient can be controlled on regions A and B.

the distortion of the arc remains between 1 and $1 + \varepsilon/2$. Hence the change in distortion of the arc is still smaller than ε. This takes care of all points in region C in Figure 2.5.

Now let us turn our attention to the subarc $\gamma([\hat{c}, \hat{d}]) \subset \gamma((c, d))$. We define $[\hat{c}, \hat{d}]$ to be any interval contained in (c, d) so that $\gamma([\hat{c}, \hat{d}])$ has positive curvature. By Theorem 36 on page 20 there exists a Radon measure $\mathcal{K}$ so that $\mathcal{K}((p, q)) = \kappa_{(p,q)}$ for all intervals $(p, q) \subset (c, d)$. We will use $\mathcal{K}$ to show that we can find a set $[\hat{c}, \hat{d}]$ with the desired properties. Since $\mathcal{K}$ is a Radon measure, if we let H denote the set of all compact subsets of (c, d) then

$$\mathcal{K}((c, d)) = \sup_{h \in H} \mathcal{K}(h)$$

Hence there must exist some sequence of compact sets h_i so that $\mathcal{K}(h_i) \to \mathcal{K}((c,d))$ as $i \to \infty$ and there must also be some j so that $\mathcal{K}(a_j) > 0$. Let $[\hat{c}, \hat{d}] \subset (c,d)$ be any closed interval containing h_i. Notice that we can replace $\gamma[\hat{c}, \hat{d}]$ with a polygon without increasing the distortion for points in C_1, C_2, and C_3 in Figure 2.5 since each of these regions is contained within the larger region C.

For given $\varepsilon' > 0$, let $P\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve that consists of two types of arcs. The first is any polygon inscribed inside the arc $\gamma([\hat{c}, \hat{d}])$ such that $\|\gamma(s) - P(s)\| < \varepsilon'$ for all $s \in [\hat{c}, \hat{d}]$, $P(\hat{c}) = \gamma(\hat{c})$, and $P(\hat{d}) = \gamma(\hat{d})$. The second arc satisfies the relation $P(s) = \gamma(s)$ for all $s \in [a,b] \setminus [\hat{c}, \hat{d}]$. Even if the arc $\gamma([\hat{c}, \hat{d}])$ is already a polygon, we can replace $\gamma([\hat{c}, \hat{d}])$ with $P([\hat{c}, \hat{d}])$ so as to decrease length.

It remains to show that we can make $d(\gamma(x), P(x); \mathbb{R}^3)$ small enough on $[\hat{c}, \hat{d}]$ so that

$$dq_P(s,t) - dq_\gamma(s,t) < \varepsilon \text{ for all pairs } (s,t) \in A \cup B$$

where A and B are defined in Figure 2.5.

Notice that if $dq_P(s,t) < dq_\gamma(s,t)$ then there is nothing to show, since we are only concerned with *increasing* distortion. So we will assume that $dq_P(s,t) > dq_\gamma(s,t)$. Suppose that $(s,t) \in A$. Then,

$$dq_P(s,t) - dq_\gamma(s,t) = \frac{d(P(s), P(t); P)}{d(P(s), P(t); \mathbb{R}^3)} - \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)}$$

$$= \frac{d(\gamma(s), \gamma(t); \mathbb{R}^3)\, d(P(s), P(t); P) - d(P(s), P(t); \mathbb{R}^3)\, d(\gamma(s), \gamma(t); \gamma)}{d(P(s), P(t); \mathbb{R}^3)\, d(\gamma(s), \gamma(t); \mathbb{R}^3)}$$

But $(s,t) \in A$ and since A is bounded away from the diagonal of $[a,b] \times [a,b]$, we know that s and t must be at least $\min\{\hat{c} - c, d - \hat{d}\}$ apart. Thus, since the curves γ and P are both embeddings, we know there is a constant C so that

$$C \geq \frac{1}{d(P(s), P(t); \mathbb{R}^3)\, d(\gamma(s), \gamma(t); \mathbb{R}^3)}$$

for all $(s,t) \in A$. So,

$$\begin{aligned}\frac{dq_P(s,t) - dq_\gamma(s,t)}{C} &\leq d(\gamma(s),\gamma(t);\mathbb{R}^3)\, d(P(s),P(t);P) - d(P(s),P(t);\mathbb{R}^3)\, d(\gamma(s),\gamma(t);\gamma) \\ &\leq d(\gamma(s),\gamma(t);\mathbb{R}^3)\, d(\gamma(s),\gamma(t);\gamma) - d(P(s),P(t);\mathbb{R}^3)\, d(\gamma(s),\gamma(t);\gamma) \\ &= d(\gamma(s),\gamma(t);\gamma)\left[d(\gamma(s),\gamma(t);\mathbb{R}^3) - d(P(s),P(t);\mathbb{R}^3)\right].\end{aligned}$$

But, $d(P(t),\gamma(s);\mathbb{R}^3) \leq d(P(t),P(s);\mathbb{R}^3) + d(P(s),\gamma(s);\mathbb{R}^3)$ and by rearranging and rewriting terms $-d(P(s),P(t);\mathbb{R}^3) \leq d(P(s),\gamma(s);\mathbb{R}^3) - d(P(t),\gamma(s);\mathbb{R}^3)$. So,

$$\begin{aligned}&d(\gamma(s),\gamma(t);\mathbb{R}^3) - d(P(s),P(t);\mathbb{R}^3) \\ &\leq d(\gamma(s),P(t);\mathbb{R}^3) + d(P(t),\gamma(t);\mathbb{R}^3) + d(P(s),\gamma(s);\mathbb{R}^3) - d(P(t),\gamma(s);\mathbb{R}^3) \\ &= d(P(t),\gamma(t);\mathbb{R}^3) + d(P(s),\gamma(s);\mathbb{R}^3).\end{aligned}$$

Finally, since $s,t \in [\hat{c},\hat{d}]$, we can define P so that

$$d(P(t),\gamma(t);\mathbb{R}^3) + d(P(s),\gamma(s);\mathbb{R}^3) < \varepsilon/(Cd(\gamma(s),\gamma(t);\gamma)),$$

completing the proof in this case. □

Case 2. *Suppose there is no subarc of* $\gamma((c,d))$ *with total curvature* $0 < K < 2\cos^{-1}(1/(1+\varepsilon/2))$.

Proof. We will show that this forces $\gamma((c,d))$ to be a polygonal arc. A new technique will be required in this case.

By the Lebesgue-Radon-Nikodym theorem (Theorem 21 on page 9), we can write the curvature measure $\mathcal{K}$ as a sum of two measures λ_a and λ_s so that $\lambda_a \ll ds$ and $\lambda_s \perp ds$. We then further decompose λ_s as a sum of atomic measures $\{\mu_{x_i}\}$ and a non-atomic measure μ which is singular with respect to arclength. So we have

$$\mathcal{K}((p,q)) = \lambda_a((p,q)) + \sum_i \mu_{x_i}((p,q)) + \mu((p,q)).$$

Suppose now that $K_0 := 2\cos^{-1}(1/(1+\varepsilon/2))$ and (p,q) is an interval so that $\mathcal{K}((p,q)) > K_0$. Then $\sum_i \mu_{x_i}((p,q)) < \infty$ and so there exists a constant M so that $\sum_{i>M} \mu_{x_i}((p,q)) < K_0/2$.

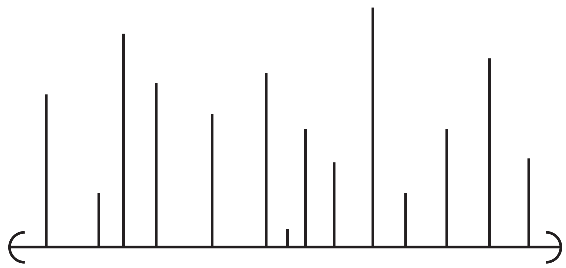

Figure 2.6: Here we see a partition of the interval (c, d) where the points that define the partition are the first M atoms in the sequence $\{x_i\}$.

Without loss of generality, assume $x_1 < \ldots < x_M$. Then we partition the interval (p, q) with the points $\{x_1, x_2, \ldots, x_M\}$ as seen in Figure 2.6.

Now, let (u, v) be a subinterval of (p, q) that lies between two adjacent partition points. Since the measure $\lambda_a + \mu$ is non-atomic, by Lemma 23 on page 9 there is a value L so that if $|v - u| < L$, then $\lambda_a((u, v)) + \mu((u, v)) < K_0/2$. Thus, we may assume that (u, v) has small enough length so that $\lambda_a((u, v)) + \mu((u, v)) < K_0/2$. Since (u, v) does not include any of the partition points $x_1, \ldots, x_M$,

$$\sum_i \mu_{x_i}((u, v)) = \sum_{i>M} \mu_{x_i}((u, v)) < K_0/2.$$

Thus $\mathcal{K}((u, v)) < K_0$ and it must be the case that $\mathcal{K}((u, v)) = 0$. Covering each interval between the M partition points $\{x_1, x_2, \ldots, x_M\}$ by overlapping open intervals with length less than $|v - u|$ shows us that $\mathcal{K}((p, q) - \{x_1, \ldots, x_M\}) = 0$. Therefore, the curve is polygonal on the interval (p, q) with corners at the atoms $\{x_1, x_2, \ldots, x_M\}$. Since this is true for any interval (p, q) it must be the case that the arc $\gamma((c, d)$ is polygonal.

It remains to describe how to decrease length without changing the distortion quotient any more than ε. First we will restrict our attention to a neighborhood of one corner small enough to guarantee that no pair of points in the neighborhood are $\mathrm{Len}(\gamma)/2$ apart along the curve γ. If the exterior angle at the corner point is ϕ, then by the calculation in the proof of Lemma 46 on page 35 the restriction of the distortion quotient to the edges meeting at the corner achieves a maximum value of $\sec\frac{\phi}{2}$ as seen in Figure 2.7.

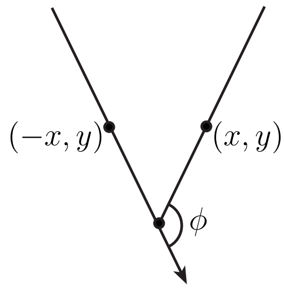

Figure 2.7: If we translate the curve so that the vertex at the corner of the arc is symmetric about the y-axis, then a calculation shows that the distortion quotient restricted to this pair of segments is maximized at points of the form (x, y) and $(-x, y)$. Moreover, this maximum is exactly equal to $\sec\frac{\phi}{2}$ where ϕ is the exterior angle at the corner. See Lemma 46 on page 35 for a formal proof.

As seen in Figure 2.8, replace the edge with vertices $(0, 0)$ and (x, y) with an edge with vertices $\epsilon(x, y)$ and (x, y) (using a small value of ϵ) and replace the edge with vertices $(-x, y)$ and $(0, 0)$ with the edge with vertices $(-x, y)$ and $\epsilon(x, y)$. The triangle inequality guarantees that this alteration will decrease length. Since $\sec(\phi/2)$ is an increasing function on the interval $(0, \pi)$, decreasing the exterior angle decreases distortion.

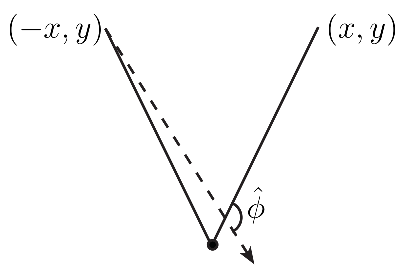

Figure 2.8: Here we shorten the length of the polygonal arc slightly while decreasing the exterior angle by a small amount by shortening one segment. This modification decreases the distortion quotient on this pair of line segments.

This completes the proof. $\square$

$\square$

2.4 Main Theorem

We now state the main theorem.

Theorem 54 (Main Theorem)**.** *Let $U_C([\gamma])$ be the set of all finite total curvature curves γ in $[\gamma]$, with distortion $\delta(\gamma) < C$ and distortion thickness $\tau_{\delta([\gamma])} \geq 1$ (i.e., any point (s,t) with $dq_\gamma(s,t) \geq \delta([\gamma])$ satisfies $\|\gamma(s) - \gamma(t)\| \geq 1$). Then any open interval on a curve of minimum length in $U_C([\gamma])$ is either a straight line segment or contains an endpoint of a $\delta([\gamma])$-drc.*

Proof. Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^3$ be a curve of minimum length in $U_C([K])$ and let (c,d) be any open interval on γ. If the total curvature along (c,d) is zero, then there is nothing to show. So assume that the total curvature along (c,d) is positive. Further assume that the interval (c,d) does *not* contain an endpoint of a $\delta([\gamma])$-drc. We will deduce a contradiction. Since there is no endpoint of a $\delta([\gamma])$-drc contained within (c,d), Proposition 52 states that there is a subinterval (p,q) of (c,d) so that if $(s,t) \in (p,q) \times [a,b]$, then $\delta([\gamma]) - dq_\gamma(s,t) > \varepsilon$ for some $\varepsilon > 0$. Provided the total curvature along (p,q) is nonzero, Proposition 53 on page 46 enables us to decrease the length of the interval (p,q) to obtain a new curve $\hat{\gamma}$ with the property that $dq_{\hat{\gamma}}(s,t) - dq_\gamma(s,t) < \varepsilon$ for $(s,t) \in (c,d) \times [a,b] \cup [a,b] \times (c,d)$. Thus no new $\delta([\gamma])$-drc's exist with endpoints in $(c,d) \times [a,b] \cup [a,b] \times (c,d)$. We claim that we have not increased the distortion quotient at any pair outside $(c,d) \times [a,b] \cup [a,b] \times (c,d)$. Indeed, if (s,t) is any point outside $(c,d) \times [a,b] \cup [a,b] \times (c,d)$, then $d(\hat{\gamma}(s), \hat{\gamma}(t); \hat{\gamma}) \leq d(\gamma(s), \gamma(t); \gamma)$ while $d(\hat{\gamma}(s), \hat{\gamma}(t); \mathbb{R}^3) = d(\gamma(s), \gamma(t); \mathbb{R}^3)$. Since $\delta(\gamma) \geq \delta([\gamma])$, the distortion of γ is realized on such a pair and this implies that $\delta(\hat{\gamma}) \leq \delta(\gamma) < C$. Therefore, $\hat{\gamma} \in U_C([\gamma])$ and $\mathrm{Len}(\hat{\gamma}) < \mathrm{Len}(\gamma)$. This is the desired contradiction. ☐

2.5 Conclusion and Open Problems

The main theorem provides a good deal of structure for distortion minimizing curves provided they have a representative of shortest length. Indeed, the frequency of drcs is highly reminiscent of the results in [2]. We hope this new information on drcs will aid in constructing better lower bounds for distortion.

We point out that the main theorem is still somewhat unsatisfactory since there may be no length-minimizing curves in $U_C([\gamma])$. Fixing this problem requires proving at least two conjectures:

Conjecture 55. *For any curve γ there exists a chord with distinct endpoints $\gamma(s)$ and $\gamma(t)$ so that $dq_\gamma(s,t) \geq \delta([\gamma])$. Hence we can rescale any knot in $[\gamma]$ to have $\delta([\gamma])$-distortion thickness 1.*

This conjecture means that the set $U_C([\gamma])$ is nonempty for all $C > \delta([\gamma])$. In this case we further conjecture:

Conjecture 56. *The set $U_C([\gamma])$ contains a minimizer for length for any knot type $[\gamma]$ and any $C > \delta([\gamma])$.*

These conjectures would complete our proposed contribution to the study of Gromov's question. Suppose that $\{\gamma_i\}$ is a family of length-minimizing knots in $U_{2\delta([\gamma_i])}([\gamma_i])$ and suppose that studying the geometry of the knots $\{\gamma_i\}$ led to finding a divergent sequence of lower bounds on $\delta(\gamma_i) \leq 2\delta([\gamma_i])$. Then by dividing these lower bounds by 2 we obtain a sequence of lower bounds on $\delta([\gamma_i])$ which is also divergent. Provided these lower bounds become arbitrarily large, we will have answered Gromov's question.

BIBLIOGRAPHY

[1] Robert B. Kusner and John M. Sullivan. *On Distortion and Thickness of Knots*, Topology and Geometry in Polymer Science (S. Whittington, D. Summers, and T. Lodge, eds.), IMA Volumes in Mathematics and its Applications, 103, Springer, 1997, arXiv:dg-ga/9702001, pp. 67-78, Proceedings of the IMA workshop, June 1996.

[2] Jason Cantarella, Joe Fu, Rob Kusner, John M. Sullivan, and Nancy Wrinkle, *Criticality for the Gehring Link Problem*, 2004, arXiv:math.DG/0402212, preprint.

[3] Elizabeth Denne, *Alternating Quadrisecants of Knots*, Ph.D. thesis, Univ. Illinois, Urbana, 2004.

[4] Mikhael Gromov, *Filling Riemannian Manifolds*, J. Diff. Geom. **18**(1983), 1-147.

[5] John W. Milnor, *On the Total Curvature of Knots*, Ann. of Math. **52**(1950), 248-257.

[6] Jun O'Hara, *Energy of a Knot*, Topology **30**(1991), 241-247.

[7] Jun O'Hara, *Family of Energy Functionals of Knots*, Topology Appl. **48**(1992), 147-161.

[8] Jun O'Hara, *Energy Functionals of Knots II*, Topology Appl. **56**(1994), 45-61.

[9] Elizabeth Denne, Yuanan Diao, and John M. Sullivan, *Quadrisecants Give New Lower Bound for the Ropelength of a Knot*, 2004, arXiv:math.DG/0408026, preprint.

[10] Elizabeth Denne, J.M. Sullivan, *The Distortion of a Knotted Curve*, arXiv:math.GT/0409438.

[11] Jason Cantarella, Greg Kuperberg, Robert B. Kusner, and John M. Sullivan, *The Second Hull of a Knotted Curve*, American Journal of Mathematics 125 (2003) no. 6, p. 1335-1348.

[12] Allen Hatcher, Algebraic Topology, Cambridge University Press, 2002.

[13] Lawrence C. Evans, Ronald F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.

[14] P.J.M. van Laarhoven and E.H.L Aarts, Simmulated Annealing: Theory and Applications, D. Reidel Publishing Company, 1987.

[15] Aaron Abrams, Jason Cantarella, Joe Fu, Mohammad Ghomi, and Ralph Howard, *Circles Minimize Most Knot Energies*, Topology 42(2002) no. 2, p.381-394.

[16] H. L. Royden, Real Analysis, Third Edition, Prentice Hall, 1988.

[17] Gerald B. Folland, Real Analysis Modern Techniques and Their Applications, Second Edition, John Wiley & Sons, Inc., 1999.

[18] Walter Rudin, Real & Complex Analysis, Third Edition, McGraw-Hill Inc., 1987.

[19] S. S. Chern, *Curves and Surfaces in Euclidean Space*, Studies in Global Geometry and Analysis, pg. 16-56, Math. Assoc. Amer., distributed by Prentice-Hall, Englewood Cliffs, N.J., 1967.

[20] C. Jordan, Cours d'analyse , Gauthier-Villars (1883)

[21] Online Encyclopedia of Mathematics, http://eom.springer.de/R/r080130.htm, This text originally appeared in Encyclopedia of Mathematics - ISBN 1402006098, edited by Michiel Hazewinkel, Springer-Verlag Berlin Heidelberg New York, 2002.

[22] John M. Sullivan, *Curves of Finite Total Curvature*, 2006, arXiv:math.GT/0606007 v1, preprint.